NORTH CAROLINA STATE UNIVERSITY AT RALEIGH

SCHOOL OF ENGINEERING

DEPARTMENT OF ENGINEERING SCIENCE AND MECHANICS
BOX 5130, ZIP 27607

October 29, 1974

Dr. Robert G. Thompson
Technical Representative of the
Contracting Officer (TRCO)
Mail Stop 230
NASA-Langley Research Center
Hampton, Virginia 23665

RE: Final Report, NASA Research Grant NGL 34-002-084

Dear Bob:

How are you?

I am enclosing two (2) copies of the Final Report of the above Research Grant. Three (3) additional copies have been forwarded directly to:

NASA Scientific and Technical
Information Facility
Post Office Box 33
College Park, Maryland 20740

This report is subdivided into three sections as follows:

Section I Journal Papers and Technical Reports of Research Projects directly related to and partially supported by the research grant.

I.1. "Stress Waves Resulting from Hypervelocity Impact," AIAA paper No. 69-355, Presented at the 1969 AIAA Hypervelocity Impact Conference in Cincinnati, Ohio (Authored by R. Madden and T. S. Chang) Results from a numerical scheme based on the method of characteristics are presented for the axially symmetric, hypervelocity impact of similar materials. The analysis is restricted to the early stages of the impact of a right circular cylinder on a halfspace. The resulting rarefaction and shock waves produced by the impact are considered as discrete wavefronts which divide the impacted zone into regions. Numerical diffusion is then controlled by requiring that the values of the dependent variables at a given point in the impacted zone to depend on only the calculated values at earlier times at points in the same region as the point in question. The numerical results give

accurate representations of the stress wave profiles (i.e., rarefaction and shock waves) which should be useful as inputs for a later stage elastoplastic analysis and/or spallation analysis. The effects of "numerical diffusion" on the calculated pressure and flow fields when the rarefaction wave is not considered as discrete is investigated and the "diffused" results are compared with the more exact analysis.

I.2. "Nonlinear Waves in a Rate-Sensitive, Elastoplastic Material," International Journal of Engineering Science, Volume 10, pp. 353-367, 1972. (Authored by E. E. Burniston and T. S. Chang).
Two classes of closed form solutions of one-dimensional, nonlinear waves in a rate-sensitive, elastoplastic material are reported. One class of these solutions is self-similar and the other class consists of constant speed propagations. Applications of these solutions to unsteady motions behind propagating discontinuities are also considered.

I.3. "Curved Characteristics Behind Blast Waves," The Physics of Fluids, Volume 15, pp. 502-504, 1972. (Authored by O. Laporte and T. S. Chang)
Exact solutions, expressed in closed form in terms of elementary functions, are presented for the three sets of curved characteristics behind a self-similar, strong blast wave.

I.4. "On Dispersion and Characteristic Motions of Temperature-Rate Dependent Materials," National Aeronautics and Space Administration Report CR-1795, 1971.
A general three-dimensional theory of a thermomechanical material which can be a metallic or polymeric medium, or a structured composite, is developed using the modern techniques of axiomatic continuum mechanics and the laws of thermodynamics. One-dimensional linear spatial gradient temperature-rate dependent theories are presented for both thermoviscoelastic and thermoelastic materials. The characteristic motions are considered and it is shown that, due to the presence of temperature-rate effects, thermal propagation speeds have finite values. A comprehensive study of the dispersion relations is presented and illustrated graphically for typical values of the material constants. Analytical expressions are obtained for both high and low frequency responses. It is demonstrated that the characteristic speeds coincide with the high frequency asymptotic phase velocities in both cases. Physical and numerical limitations on the material constants are obtained for stable wave propagations. A class of self-similar solutions is obtained for the temperature-rate dependent thermoelastic medium using the theory of continuous group of transformations.

I.5. "Comments on Application of Singular Eigenfunction Expansions to the Propagation of Periodic Disturbances in a Radiating Grey Gas," The Physics of Fluids, Volume 16, pp. 159-160, 1973. (Authored by T. S. Chang, K. H. Kim and M. N. Osizik)
Recently, we have been extending our analysis of sound propagation

in dissociative, radiative media to include the effects of scattering. It is of interest to note that the set of pertinent integrodifferential equations can also be solved exactly using the singular eigenfunction expansion technique.

Section II Computer Programs and Numerical Results of Hypervelocity Impact Calculations

II.1. Method of Characteristics Calculations and the Computer Code Using Orthogonal Polynomial Least Square Surface Fits.
Very recently, we have developed a new numerical scheme using the method of characteristics to calculate the flow properties and pressures behind decaying shock waves for materials under hypervelocity impact. This procedure is quite different from our earlier methods (see paper I.1). We are now able to replace the time-consuming double interpolation subroutines used in paper I.1 by a technique based on Orthogonal Polynomial Least Square Surface Fits. Typical calculated results are given and in Tables 1-19. These results are compared with the double-interpolation results. The complete computer program is also included.

II.2. Method of Characteristics Calculations and the Computer Code for Materials with Arbitrary Equations of State.
As reported in our earlier status reports, we have developed a numerical code capable of calculating flow properties and pressures behind decaying shock waves for materials under hypervelocity impact with arbitrary equations of state. For comparison purposes, we have made some sample calculations using the new code for impact conditions similar to those reported in I.1. These results are quite encouraging and are displayed graphically in Figures 1-3. A listing of the numerical code is also attached.

Section III Fundamental Research on Equations of State and Non-Equilibrium Statistical Mechanics Generated by the Research Grant.

Materials under hypervelocity impact experience extreme changes of stresses, strains, and thermodynamic states. We have looked into some basic research areas related to (1) the non-equilibrium statistical mechanical aspects of heat transfer with discontinuous velocity boundary conditions, and (2) the fundamental understanding of the equations of state, phase transitions, and critical phenomena of materials under extreme thermodynamic environments. Our research efforts along these directions are found in the following research papers:

III.1. "Elementary Solutions of Coupled Model Equations in the Kinetic Theory of Gases," International Journal of Engineering Science, Volume 12, pp. 441-470, 1974. (Authors: J. T. Kriese, T. S. Chang, and C. E. Siewert.)

The method of elementary solutions is employed to solve two coupled integrodifferential equations sufficient for determining temperature - density effects in a linearized BGK model in the kinetic theory of gases. Full-range completeness and orthogonality theorems are proved for the developed normal modes and the infinite-medium Green's function is constructed as an illustration of the full-range formalism. The appropriate homogeneous matrix Riemann problem is discussed, and half-range completeness and orthogonality theorems are proved for a certain subset of the normal modes. The required existence and uniqueness theorems relevant to the H matrix, basic to the half-range analysis, are proved, and an accurate and efficient computational method is discussed. The half-space temperature-slip problem is solved analytically, and a highly accurate value of the temperature-slip coefficient is reported.

III.2. "Tricritical Points in Multicomponent Fluid Mixtures," Physical Review, Volume A9, pp. 2573-2578, 1974. (Authors: A. Hankey, T. S. Chang, and H. E. Stanley)
In view of experimental considerations, we give a model-independent argument that the novel tricritical points in multicomponent fluid mixtures, where three phases simultaneously become critical, are points on the boundary of a single two-dimensional surface of critical points. This result is corroborated by the Landau model suggested by Griffiths. The relationship between these tricritical points and the complex "higher-order" critical points proposed to exist in certain magnetic systems is elucidated.

III.3. "Generalized Scaling Hypothesis in Multicomponent Systems. I. Classification of Critical Points by Order and Scaling at Tricritical Points," Physical Review, Volume B8, pp. 346-364. (Authored by T. S. Chang, A. Hankey, and H. E. Stanley.)
The goal of this work is to provide an analysis of spaces of critical points for multicomponent systems. First, we propose the geometric concept of order O for critical points; we distinguish it from a previous definition of a "multicritical" point. Specifically, we may define the intersection of space of critical points of order O to be a space of critical points of order $(O + 1)$. Ordinary critical points are defined to be of order $O = 2$, so that the tricritical points introduced by Griffiths are of order $O = 3$. We discuss more general examples of critical spaces of order $O = 3$ which are known for a wide variety of systems; we also propose several examples of models of magnetic systems showing critical points of order $O = 4$ - i.e., systems having intersecting lines of tricritical points. The analysis of critical and coexistence spaces also provides a new form of the Gibbs phase rule suitable for complex magnetic models. Next we define - for the critical points of order O of which examples have been given - special directions in terms of which to make a scaling hypothesis. We give the hypothesis for simple systems and then for tricritical points, and then, in a subsequent paper, part II, the special directions are used to make a scaling hypothesis at spaces of critical points of any order. Certain predictions (e.g., scaling laws and "single-power"

scaling functions) follow in a simple and straightforward fashion. We consider the scaling hypothesis at a critical space of order O in terms of a group of transformations. We can define a set of invariants of the group. It is possible, for $O > 3$, to make a second scaling hypothesis for the space of order $O - 1$ using certain of these invariants as independent variables. This is advantageous because certain "double-power" scaling functions then follow directly; these predict that for $O = 3$, experimental data collapse from a volume onto a line. This prediction is to be contrasted with ordinary scaling function, which predict that data collapse by only a single dimension (e.g., from a volume onto a surface or from a surface onto a line).

III.4. "Double-Power Scaling Functions Near Tricritical Points," Physical Review, Volume B7, pp. 4263-4266, 1973. (Authored by T. S. Chang, A. Hankey, and H. E. Stanley.)
We introduce invariants of the scaling equation about the tricritical point. Using these invariants, a modified version of the scaling hypothesis about the three critical lines meeting at the tricritical point is presented. From it we demonstrate that the thermodynamic equation of state near a tricritical point and near a critical line may be expressed as double-power scaling functions. These imply that experimental data should collapse from a volume onto a line (i.e., by two dimensions). This behavior is in contrast to ordinary "single-power" scaling functions, which predict data collapsing from a volume onto a surface or from a surface onto a line (i.e., by one dimension).

In conclusion, we have obtained much information pertaining to the behavior materials under impact, particularly hypervelocity impact. Two useful numerical des have been developed for calculations of axisymmetric hypervelocity impact for terials with rather general equations of state. Fundamental research in the eas of critical equations of state and non-equilibrium statistical mechanics and at transfer have also been considered.

With best wishes, I am

Very sincerely yours,

Tom

Tien Sun Chang
Professor

C:mlm

closures

: NASA Scientific and Technical
Information Facility (3 copies)

I.1. "Stress Waves Resulting from Hypervelocity Impact"

I.2. "Nonlinear Waves in a Rate-Sensitive, Elastoplastic Material"

I.3. "Curved Characteristics Behind Blast Waves"

I.4. "On Dispersion and Characteristic Motions of Temperature-Rate Dependent Materials"
NASA Report - copies not enclosed

I.5. "Comments on Application of Singular Eigenfunction Expansions to the Propagation of Periodic Disturbances in a Radiating Grey Gas"

PAPERS INTENTIONALLY OMITTED

II.1. Method of Characteristics Calculations and the Computer Code Using Orthogonal Polynomial Least Square Surface Fits

List of Symbols

t = time (μsec)

r = radial coordinate (cm)

Z = axial coordinate (cm)

U = radial velocity (cm/μsec)

V = axial velocity (cm/μsec)

ρ = mass density (gm/cm^3)

e = internal energy per unit mass (megabar cm^3/gm)

a = sound speed (cm/μsec)

P = pressure (megabars)

$\cos \alpha_o$ = direction cosine of normal with respect to r axis

$\sin \alpha_o$ = direction cosine of normal with respect to Z axis

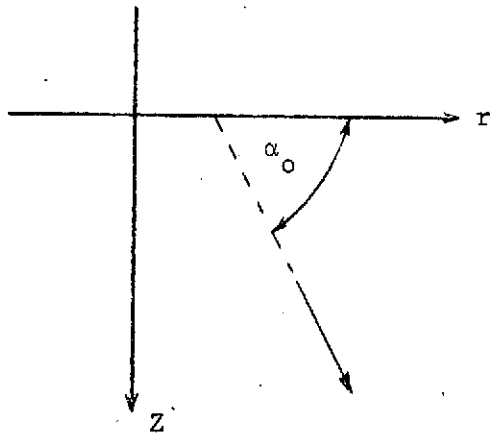

Table 1. Flow properties in Projectile Shock at t = 1.18 μsec after impact of a 2.5-cm-diameter projectile at 0.76 cm/μsec on an aluminum half-space based on (1) Double Interpolation and (2) Othogonal Polynomials Least Square Surface Fit with Double Interpolation after t = 1.1 μsec.

(1)					(2)				
Z ρ	r e	P a	U cos α_o	V sin α_o	Z ρ	r e	P a	U cos α_o	V sin α_o
-0.3532	0.0	1.0869	0.0	0.3800	-0.3532	0.0	1.0869	0.0	0.3800
4.2103	0.0722	1.0038	0.0	1.0000	4.2103	0.0722	1.0038	0.0	1.0000
-0.3532	0.2500	1.0869	0.0	0.3800	-0.3532	0.2500	1.0869	0.0	0.3800
4.2103	0.0722	1.0038	0.0	1.0000	4.2103	0.0722	1.0038	0.0	1.0000
-0.3521	0.5012	1.0412	0.0025	0.3910	-0.3509	0.5009	0.9911	0.0017	0.4033
4.1738	0.0681	0.9934	-0.0068	1.0000	4.1325	0.0636	0.9820	-0.0049	1.0000
-0.3312	0.7611	0.8591	0.0209	0.4372	-0.3312	0.7630	0.7891	0.0106	0.4554
4.0185	0.0521	0.9500	-0.0648	0.9976	3.9533	0.0463	0.9318	-0.0348	0.9991
-0.2120	1.0183	0.4162	0.0124	0.5679	-0.2117	1.0177	0.4090	0.0116	0.5704
3.5399	0.0183	0.8122	-0.0667	0.9968	3.5303	0.0178	0.8094	-0.0613	0.9972

Table 2. Flow properties in Target Shock at t = 1.18 μsec after impact of a 2.5-cm-diameter projectile at 0.76 cm/μsec on an aluminum half-space based on (1) Double Interpolation and (2) Orthogonal Polynomials Least Square Surface Fit with Double Interpolation after t = 1.1 μsec.

(1)					(2)				
Z ρ	r e	P a	U cos α_o	V sin α_o	Z ρ	r e	P a	U cos α_o	V sin α_o
1.2500	0.0	1.0869	0.0	0.3800	1.2500	0.0	1.0869	0.0	0.3800
4.2103	0.0722	1.0038	0.0	1.0000	4.2103	0.0722	1.0038	0.0	1.0000
1.2500	0.2500	1.0869	0.0	0.3800	1.2500	0.2500	1.0869	0.0	0.3800
4.2103	0.0722	1.0038	0.0	1.0000	4.2103	0.0722	1.0038	0.0	1.0000
1.2500	0.5007	1.0686	0.0029	0.3756	1.2497	0.5007	1.0685	0.0029	0.3756
4.1957	0.0750	0.9997	0.0078	1.0000	4.1957	0.0705	0.9997	0.0078	1.0000
1.2398	0.7544	0.9647	0.0058	0.3501	1.2398	0.7544	0.9647	0.0058	0.3501
4.1106	0.0613	0.9758	0.0165	0.9998	4.1106	0.0613	0.9758	0.0165	0.9998
1.2143	1.0281	0.8382	0.0192	0.3169	1.2143	1.0281	0.8382	0.0192	0.3169
3.9996	0.0503	0.9446	0.0605	0.9979	3.9996	0.0503	0.9446	0.0605	0.9979
1.1235	1.3139	0.5624	0.0302	0.2368	1.1236	1.3139	0.5679	0.0304	0.2384
3.7190	0.0285	0.8649	0.1265	0.9910	3.7252	0.0290	0.8666	0.1265	0.9910
0.7683	1.7372	0.1558	0.0540	0.0691	0.7678	1.7377	0.1546	0.0528	0.0694
3.1170	0.0038	0.6776	0.6152	0.7870	3.1144	0.0037	0.6768	0.6050	0.7948
0.5808	1.8634	0.0670	0.0330	0.0263	0.5807	1.8634	0.0664	0.0327	0.0261
2.9088	0.0008	0.6045	0.7817	0.6227	2.9071	0.0008	0.6039	0.7816	0.6227
0.3716	1.9378	0.0	0.0	0.0	0.3716	1.9377	0.0	0.0	0.0
2.7000	0.0	0.5277	0.9289	0.3690	2.7000	0.0	0.5277	0.9289	0.3690

Table 3. Flow properties in Rarefaction at t = 1.18 μsec after impact of a 2.5-cm-diameter projectile at 0.76 cm/μsec on an aluminum half-space based on (1) Double Interpolation and (2) Orthogonal Polynomials Least Square Surface Fit with Double Interpolation after t = 1.1 μsec.

(1)				(2)			
Z	r	$\cos \alpha_o$	$\sin \alpha_o$	Z	r	$\cos \alpha_o$	$\sin \alpha_o$
1.2500	0.3781	0.6768	-0.7362	1.2500	0.3781	0.6768	-0.7362
1.1356	0.2852	0.5801	-0.8145	1.1355	0.2852	0.5801	-0.8145
1.0210	0.2132	0.4834	-0.8754	1.0210	0.2132	0.4834	-0.8754
0.9065	0.1577	0.3867	-0.9222	0.9065	0.1577	0.3867	-0.9222
0.7920	0.1165	0.2901	-0.9570	0.7920	0.1165	0.2901	-0.9570
0.6774	0.0879	0.1934	-0.9811	0.6774	0.0879	0.1934	-0.9811
0.5629	0.0711	0.0967	-0.9953	0.5629	0.0711	0.0967	-0.9953
0.4484	0.0656	-0.0000	-1.0000	0.4484	0.0656	-0.0000	-1.0000
0.3339	0.0711	-0.0967	-0.9953	0.3339	0.0711	-0.0967	-0.9953
0.2194	0.0879	-0.1934	-0.9811	0.2194	0.0879	-0.1934	-0.9811
0.1048	0.1165	-0.2901	-0.9570	0.1048	0.1165	-0.2901	-0.9570
-0.0097	0.1577	-0.3867	-0.9222	-0.0097	0.1577	-0.3867	-0.9222
-0.1242	0.2132	-0.4834	-0.8754	-0.1242	0.2132	-0.4834	-0.8754
-0.2387	0.2852	-0.5801	-0.8145	-0.2387	0.2852	-0.5801	-0.8145
-0.3532	0.3781	-0.6768	-0.7362	-0.3532	0.3781	-0.6768	-0.7362

Table 4. Flow properties in Interior Region at t = 1.18 μsec after impact of a 2.5-cm-diameter projectile at 0.76 cm/μsec on an aluminum half-space based on (1) Double Interpolation and (2) Orthogonal Polynomial Least Square Surfact Fit with Double Interpolation after t = 1.1 μsec at Z = -0.25 cm.

Z = - 0.25 cm	r	P	U	V	ρ	e	a
(1)	0.0	1.0869	0.0	0.3800	4.2103	0.0722	1.0038
	0.2500	1.0869	0.0	0.3800	4.2103	0.0722	1.0038
	0.5000	0.9289	0.0167	0.4159	4.050	0.0622	0.9654
	0.7500	0.6891	0.0558	0.3987	3.8200	0.0422	0.9012
	1.0000	0.0	0.0	0.0	2.7000	0.0	0.5277
	1.2500	0.0	0.0	0.0	2.7000	0.0	0.5277
	1.5000	0.0	0.0	0.0	2.7000	0.0	0.5277
	1.7500	0.0	0.0	0.0	2.7000	0.0	0.5277
	2.0000	0.0	0.0	0.0	2.7000	0.0	0.5277
(2)	0.0	1.0869	0.0	0.3800	4.2103	0.0722	1.0038
	0.2500	1.0869	0.0	0.3800	4.2103	0.0722	1.0038
	0.5000	0.6710	-0.1555	0.4333	3.7854	0.0425	0.8946
	0.7500	2.2439	-0.0865	0.9374	5.2602	0.1201	1.2174
	1.0000	0.0	0.0	0.0	2.7000	0.0	0.5277
	1.2500	0.0	0.0	0.0	2.7000	0.0	0.5277
	1.5000	0.0	0.0	0.0	2.7000	0.0	0.5277
	1.7500	0.0	0.0	0.0	2.7000	0.0	0.5277
	2.0000	0.0	0.0	0.0	2.7000	0.0	0.5277

Table 5. Flow properties in Interior Region at t = 1.18 μsec after impact of a 2.5-cm-diameter projectile at 0.76 cm/μsec on an aluminum half-space based on (1) Double Interpolation and (2) Orthogonal Polynomial Least Square Surface Fit with Double Interpolation after t = 1.1 μsec at Z = 0.0 cm.

Z = 0.0 cm	r	P	U	V	ρ	e	a
(1)	0.0	1.0869	0.0	0.3800	4.2103	0.0722	1.0038
	0.2500	0.9644	0.0130	0.3829	4.0861	0.0646	0.9743
	0.5000	0.6875	0.0985	0.4466	3.7705	0.0478	0.8972
	0.7500	0.4822	0.2065	0.5931	3.5022	0.0348	0.8269
	1.0000	0.1931	0.2829	0.7128	3.0774	0.0132	0.6916
	1.2500	0.0	0.6160	0.3904	2.7000	0.0	0.5277
	1.5000	0.0	0.1923	-0.0971	2.7000	0.0	0.5277
	1.7500	0.0	0.1103	-0.1452	2.7000	0.0	0.5277
	2.0000	0.0	0.0303	-0.0074	2.7000	0.0	0.5277
(2)	0.0	1.0869	0.0	0.3800	4.2103	0.0722	1.0038
	0.2500	0.4956	-0.1726	0.3333	3.5563	0.0318	0.8347
	0.5000	0.5533	0.0033	0.4539	3.6020	0.0395	0.8537
	0.7500	0.7227	0.2308	0.6779	3.8156	0.0497	0.9081
	1.0000	0.0112	0.3021	0.6287	2.6397	0.0053	0.5362
	1.2500	0.0	0.3460	0.4024	2.7000	0.0	0.5277
	1.5000	0.0	0.2988	-0.1707	2.7000	0.0	0.5277
	1.7500	0.0	0.0076	-0.0333	2.7000	0.0	0.5277
	2.000	0.0	1.0304	0.0920	2.7000	0.0	0.5277

Table 6. Flow properties in Interior Region at t = 1.18 μsec after impact of a 2.5-cm-diameter projectile at 0.76 cm/μsec on an aluminum half-space based on (1) Double Interpolation and (2) Orthogonal Polynomial Least Square Surface Fit with Double Interpolation after t = 1.1 μsec at Z = 0.25 cm.

Z = 0.25 cm	r	P	U	V	ρ	e	a
(1)	0.0	1.0869	0.0	0.3800	4.2103	0.0722	1.0038
	0.2500	0.9018	0.0414	0.3889	4.0190	0.0608	0.9582
	0.5000	0.6611	0.1029	0.3898	3.7383	0.0461	0.8890
	0.7500	0.5468	0.1589	0.4410	3.5875	0.0393	0.8504
	1.0000	0.4078	0.2552	0.5698	3.3982	0.0296	0.7973
	1.2500	0.2713	0.3356	0.2505	3.1741	0.0216	0.7328
	1.5000	0.1800	0.2782	-0.0038	3.0497	0.0126	0.6831
	1.7500	0.1452	0.1345	-0.0307	3.0979	0.0032	0.6704
	2.0000	0.0	0.0	0.0	2.7000	0.0	0.5277
(2)	0.0	1.0869	0.0	0.3800	4.2103	0.0722	1.0038
	0.2500	0.9041	0.0405	0.4685	4.0212	0.0610	0.9588
	0.5000	0.6238	0.1083	0.3428	3.6922	0.0436	0.8770
	0.7500	0.4999	0.1591	0.4221	3.5219	0.0365	0.8332
	1.0000	0.4384	0.1553	0.5946	3.4452	0.0314	0.8102
	1.2500	0.3310	0.3414	0.2448	3.2767	0.0250	0.7629
	1.5000	0.1820	0.2935	0.0031	3.0507	0.0130	0.6841
	1.7500	0.1509	0.1379	-0.0592	3.1113	0.0033	0.6746
	2.0000	0.0	0.0	0.0	2.7000	0.0	0.5277

Table 7. Flow properties in Interior Region at t = 1.18 μsec after impact of a 2.5-cm-diameter projectile at 0.76 cm/μsec on an aluminum half-space based on (1) Double Interpolation and (2) Orthogonal Polynomial Least Square Surface Fit with Double Interpolation after t = 1.1 μsec at Z = 0.50 cm.

Z = 0.50 cm	r	P	U	V	ρ	e	a
(1)	0.0	1.0869	0.0	0.3800	4.2103	0.0722	1.0038
	0.2500	0.9285	0.0338	0.3758	4.0475	0.0625	0.9651
	0.5000	0.7314	0.0768	0.3523	3.8244	0.0504	0.9106
	0.7500	0.6101	0.1090	0.3385	3.6714	0.0432	0.8723
	1.0000	0.5674	0.1441	0.3850	3.6154	0.0406	0.8577
	1.2500	0.3240	0.2039	0.1874	3.2650	0.0246	0.7595
	1.5000	0.2336	0.1796	0.0690	3.2048	0.0115	0.7206
	1.7500	0.1461	0.0865	0.0345	3.1018	0.0031	0.6713
	2.0000	0.0	0.0	0.0	2.7000	0.0	0.5277
(2)	0.0	1.0869	0.0	0.3800	4.2103	0.0722	1.0038
	0.2500	0.9477	0.0025	0.4235	4.0676	0.0637	0.9700
	0.5000	0.7109	0.0920	0.3266	3.8004	0.0491	0.9045
	0.7500	0.5934	0.0993	0.3379	3.6493	0.0422	0.8666
	1.0000	0.5876	0.1442	0.3938	3.6421	0.0418	0.8647
	1.2500	0.3056	0.2154	0.1785	3.2283	0.0241	0.7501
	1.5000	0.2380	0.1708	0.0771	3.2140	0.0117	0.7232
	1.7500	0.1611	0.1319	0.0292	3.1346	0.0036	0.6820
	2.000	0.0	0.0	0.0	2.7000	0.0	0.5277

Table 8. Flow properties in Interior Region at t = 1.18 μsec after impact of a 2.5-cm-diameter projectile at 0.76 cm/μsec on an aluminum half-space based on (1) Double Interpolation and (2) Orthogonal Polynomial Least Square Surface Fit with Double Interpolation after t = 1.1 μsec at Z = 0.75 cm.

Z = 0.75 cm	r	P	U	V	ρ	e	a
(1)	0.0	1.0869	0.0	0.3800	4.2103	0.0722	1.0038
	0.2500	1.0006	0.0200	0.3730	4.1231	0.0669	0.9832
	0.5000	0.8307	0.0490	0.3472	3.9397	0.0565	0.9391
	0.7500	0.7051	0.0710	0.3163	3.7917	0.0489	0.9026
	1.0000	0.6166	0.0937	0.3127	3.6908	0.0423	0.8752
	1.2500	0.4242	0.1292	0.1985	3.4661	0.0263	0.8077
	1.5000	0.3014	0.1187	0.1145	3.3803	0.0108	0.7624
	1.7500	0.0	0.0	0.0	2.7000	0.0	0.5277
	2.0000	0.0	0.0	0.0	2.7000	0.0	0.5277
(2)	0.0	1.0869	0.0	0.3800	4.2103	0.0722	1.0038
	0.2500	0.9148	-0.0201	0.4072	4.0334	0.0615	0.9616
	0.5000	0.8167	0.0542	0.3333	3.9244	0.0556	0.9352
	0.7500	0.6996	0.0658	0.3149	3.7846	0.0486	0.9009
	1.0000	0.5993	0.0982	0.3095	3.6667	0.0414	0.8693
	1.2500	0.4229	0.1272	0.1785	3.4620	0.0265	0.8070
	1.5000	0.3248	0.1134	0.1264	3.4247	0.0115	0.7744
	1.7500	0.0	0.0	0.0	2.7000	0.0	0.5277
	2.0000	0.0	0.0	0.0	2.7000	0.0	0.5277

Table 9. Flow properties in Interior Region at t = 1.18 μsec after impact of a 2.5-cm-diameter projectile at 0.76 cm/μsec on an aluminum half-space based on (1) Double Interpolation and (2) Orthogonal Polynomial Least Square Surface Fit with Double Interpolation after t = 1.1 μsec at Z = 1.00 cm.

Z = 1.00 cm	r	P	U	V	ρ	e	a
(1)	0.0	1.0869	0.0	0.3800	4.2103	0.0722	1.0038
	0.2500	1.0869	0.0	0.3800	4.2103	0.0722	1.0038
	0.5000	0.9551	0.0239	0.3569	4.0754	0.0642	0.9719
	0.7500	0.8458	0.0404	0.3263	3.9658	0.0562	0.9438
	1.0000	0.7492	0.0545	0.3053	3.8760	0.0477	0.9180
	1.2500	0.5566	0.0703	0.2330	3.6866	0.0306	0.8607
	1.5000	0.0	0.0	0.0	2.7000	0.0	0.5277
	1.7500	0.0	0.0	0.0	2.7000	0.0	0.5277
	2.0000	0.0	0.0	0.0	2.7000	0.0	0.5277
(2)	0.0	1.0869	0.0	0.3800	4.2103	0.0722	1.0038
	0.2500	1.0869	0.0	0.3800	4.2103	0.0722	1.0038
	0.5000	0.9137	0.0337	0.3572	4.0315	0.0616	0.9613
	0.7500	0.8233	0.0460	0.3269	3.9397	0.0549	0.9376
	1.0000	0.7430	0.0646	0.3254	3.8604	0.0465	0.9137
	1.2500	0.5142	0.0880	0.2854	3.6369	0.0275	0.8463
	1.5000	0.0	0.0	0.0	2.7000	0.0	0.5277
	1.7500	0.0	0.0	0.0	2.7000	0.0	0.5277
	2.0000	0.0	0.0	0.0	2.7000	0.0	0.5277

Table 10. Flow properties in Interior Region at t = 1.18 μsec after impact of a 2.5-cm-diameter projectile at 0.76 cm/μsec on an aluminum half-space based on (1) Double Interpolation and (2) Orthogonal Polynomial Least Square Surface Fit with Double Interpolation after t = 1.1 μsec at Z = 1.25 cm.

Z = 1.25 cm	r	P	U	V	ρ	e	a
(1)	0.0	1.0869	0.0	0.3800	4.2103	0.0722	1.0038
	0.2500	1.0869	0.0	0.3800	4.2103	0.0722	1.0038
	0.5000	0.0	0.0	0.0	2.7000	0.0	0.5277
	0.7500	0.0	0.0	0.0	2.7000	0.0	0.5277
	1.0000	0.0	0.0	0.0	2.7000	0.0	0.5277
	1.2500	0.0	0.0	0.0	2.7000	0.0	0.5277
	1.5000	0.0	0.0	0.0	2.7000	0.0	0.5277
	1.7500	0.0	0.0	0.0	2.7000	0.0	0.5277
	2.0000	0.0	0.0	0.0	2.7000	0.0	0.5277
(2)	0.0	1.0869	0.0	0.3800	4.2103	0.0722	1.0038
	0.2500	1.0869	0.0	0.3800	4.2103	0.0722	1.0038
	0.5000	0.0	0.0	0.0	2.7000	0.0	0.5277
	0.7500	0.0	0.0	0.0	2.7000	0.0	0.5277
	1.0000	0.0	0.0	0.0	2.7000	0.0	0.5277
	1.2500	0.0	0.0	0.0	2.7000	0.0	0.5277
	1.5000	0.0	0.0	0.0	2.7000	0.0	0.5277
	1.7500	0.0	0.0	0.0	2.7000	0.0	0.5277
	2.0000	0.0	0.0	0.0	2.7000	0.0	0.5277

Table 11. Flow properties in Projectile Shock at t = 1.25 μsec after impact of a 2.5-cm-diameter projectile 0.76 cm/μsec on an aluminum half-space based on (1) Double Interpolation and (2) Orthogonal Polynomials Least Square Surface Fit with Double Interpolation after t = 1.1 μsec.

(1)					(2)				
Z ρ	r e	P a	U cos α_o	V sin α_o	Z ρ	r e	P a	U cos α_o	V sin α_o
-0.3742	0.0	1.0869	0.0	0.3800	-0.3742	0.0	1.0869	0.0	0.3800
4.2103	0.0722	1.0038	0.0	1.0000	4.2103	0.0722	1.0038	0.0	1.0000
-0.3742	0.2500	1.0869	0.0	0.3800	-0.3742	0.2500	1.0869	0.0	0.3800
4.2103	0.0722	1.0038	0.0	1.0000	4.2103	0.0722	1.0038	0.0	1.0000
-0.3713	0.5018	1.0071	0.0029	0.3993	-0.3684	0.5014	0.9280	0.0019	0.4191
4.1460	0.0650	0.9857	-0.0079	0.9999	4.0789	0.0581	0.9670	-0.0057	1.0000
-0.3468	0.7660	0.8343	0.0217	0.4438	-0.3437	0.7655	0.7182	0.0104	0.4750
3.9954	0.0501	0.9436	-0.0685	0.9973	3.8844	0.0405	0.9123	-0.0366	0.9990
-0.2153	1.0232	0.4007	0.0144	0.5731	-0.2140	1.0213	0.3767	0.0113	0.5816
3.5195	0.0173	0.8060	-0.0775	0.9958	3.4869	0.0157	0.7961	-0.0636	0.9970

Table 12. Flow properties in Target Shock at t = 1.25 μsec after impact of a 2.5-cm-diameter projectile at 0.76 cm/μsec or an aluminum half-space based on (1) Double Interpolation and (2) Orthogonal Polynomials Least Square Surface Fit with Double Interpolation after t = 1.1 μsec.

(1)					(2)				
Z ρ	r e	P a	U cos α_o	V sin α_o	Z ρ	r e	P a	U cos α_o	V sin α_o
1.3242	0.0	1.0869	0.0	0.3800	1.3242	0.0	1.0869	0.0	0.3800
4.2103	0.0722	1.0038	0.0	1.0000	4.2103	0.0722	1.0038	0.0	1.0000
1.3242	0.2500	1.0869	0.0	0.3800	1.3242	0.2500	1.0869	0.0	0.3800
4.2103	0.0722	1.0038	0.0	1.0000	4.2103	0.0722	1.0038	0.0	1.0000
1.3233	0.5015	1.0568	0.0039	0.3724	1.3232	0.5014	1.0550	0.0034	0.3723
4.1863	0.0695	0.9970	0.0105	0.9999	4.2848	0.0693	0.9966	0.0092	1.0000
1.3110	0.7558	0.9511	0.0066	0.3467	1.3109	0.7558	0.9506	0.0066	0.3465
4.0991	0.0601	0.9726	0.0190	0.9998	4.0986	0.0600	0.9725	0.0190	0.9998
1.2823	1.0325	0.8208	0.0203	0.3122	1.2823	1.0325	0.8240	0.0204	0.3130
3.9830	0.0490	0.9401	0.0650	0.9976	3.9859	0.0492	0.9409	0.0649	0.9976
1.1836	1.3220	0.5493	0.0312	0.2325	1.1840	1.3224	0.5610	0.0332	0.2359
3.7041	0.0276	0.8654	0.1328	0.9901	3.7174	0.0284	0.8645	0.1390	0.9892
0.8044	1.7655	0.1534	0.0550	0.0669	0.8039	1.7655	0.1475	0.0512	0.0664
3.1120	0.0037	0.6759	0.6341	0.7718	3.0997	0.0035	0.6717	0.6098	0.7910
0.6060	1.8958	0.0637	0.0318	0.0248	0.6060	1.8957	0.0621	0.0310	0.0242
2.9000	0.0007	0.6013	0.7876	0.6152	2.8955	0.0007	0.5997	0.7875	0.6152
0.3853	1.9721	0.0	0.0	0.0	0.3853	1.9722	0.0	0.0	0.0
2.7000	0.0	0.5277	0.9354	0.3522	2.7000	0.0	0.5277	0.9354	0.3524

Table 13. Flow properties in Rarefaction at t = 1.25 μsec after impact of a 2.5-cm-diameter projectile at 0.76 cm/μsec on an aluminum half-space based on (1) Double Interpolation and (2) Orthogonal Polynomials Least Square Surface Fit with Double Interpolation after t = 1.1 μsec.

(1)				(2)			
Z	r	$\cos \alpha_o$	$\sin \alpha_o$	Z	r	$\cos \alpha_o$	$\sin \alpha_o$
1.3242	0.3263	0.6768	-0.7362	1.3242	0.3263	0.6768	-0.7362
1.2029	0.2280	0.5801	-0.8145	1.2029	0.2280	0.5801	-0.8145
1.0816	0.1517	0.4834	-0.8754	1.0816	0.1517	0.4834	-0.8754
0.9602	0.0929	0.3867	-0.9222	0.9602	0.0929	0.3867	-0.9222
0.8389	0.0493	0.2901	-0.9570	0.8389	0.0493	0.2901	-0.9570
0.7176	0.0190	0.1934	-0.9811	0.7176	0.0190	0.1934	-0.9811
0.5963	0.0012	0.0967	-0.9953	0.5963	0.0012	0.0967	-0.9953
0.4750	-0.0047	0.0000	-1.0000	0.4750	-0.0047	0.0000	-1.0000
0.3537	0.0012	-0.0967	-0.9953	0.3537	0.0012	-0.0967	-0.9953
0.2324	0.0190	-0.1934	-0.9811	0.2324	0.0190	-0.1934	-0.9811
0.1111	0.0493	-0.2901	-0.9570	0.1111	0.0493	-0.2901	-0.9570
-0.0102	0.0929	-0.3867	-0.9222	-0.0102	0.0929	-0.3867	-0.9222
-0.1316	0.1517	-0.4834	-0.8754	-0.1316	0.1517	-0.4834	-0.8754
-0.2529	0.2280	-0.5801	-0.8145	-0.2529	0.2280	-0.5801	-0.8145
-0.3742	0.3263	-0.6768	-0.7362	-0.3742	0.3263	-0.6768	-0.7362

Table 14. Flow properties in Interior Region at t = 1.11 μsec after impact of a 2.5-cm-diameter projectile at 0.76 cm/μsec on an aluminum half-space based on (1) Double Interpolation and (2) Orthogonal Polynomials Least Square Surface Fit with Double Interpolation afte t = 1.1 μsec at Z = -0.25 cm.

Z = - 0.25cm	r	P	U	V	ρ	e	a
(1)	0.0	1.0869	0.0	0.3800	4.2103	0.0722	1.0038
	0.2500	1.0869	0.0	0.3800	4.2103	0.0722	1.0038
	0.5000	1.0000	0.0011	0.3947	4.1232	0.0668	0.9832
	0.7500	0.9432	0.0686	0.4380	4.1035	0.0579	0.9715
	1.0000	0.0	0.0	0.0	2.70000	0.0	0.5277
	1.2500	0.0	0.0	0.0	2.7000	0.0	0.5277
	1.5000	0.0	0.0	0.0	2.7000	0.0	0.5277
	1.7500	0.0	0.0	0.0	2.7000	0.0	0.5277
	2.0000	0.0	0.0	0.0	2.7000	0.0	0.5277
(2)	0.0	1.0869	0.0	0.3800	4.2103	0.0722	1.0038
	0.2500	1.0869	0.0	0.3800	4.2103	0.0722	1.0038
	0.5000	0.9865	-0.0923	0.4746	4.1090	0.0660	0.9798
	0.7500	1.6829	0.0102	0.7228	4.7963	0.0979	1.1264
	1.0000	0.0	0.0	0.0	2.7000	0.0	0.5277
	1.2500	0.0	0.0	0.0	2.7000	0.0	0.5277
	1.5000	0.0	0.0	0.0	2.7000	0.0	0.5277
	1.7500	0.0	0.0	0.0	2.7000	0.0	0.5277
	2.0000	0.0	0.0	0.0	2.7000	0.0	0.5277

Table 15. Flow properties in Interior Region at t = 1.11 μsec after impact of a 2.5-cm-diameter projectile at 0.76 cm/μsec on an aluminum half-space based on (1) Double Interpolation and (2) Orthogonal Polynomials Least Square Surface Fit with Double Interpolation after t = 1.1 μsec at Z = 0.0 cm.

Z = 0.0 cm	r	P	U	V	ρ	e	a
(1)	0.0	1.0869	0.0	0.38000	4.2103	0.0722	1.0038
	0.2500	1.0869	0.0	0.3800	4.2103	0.0722	1.0038
	0.5000	0.7294	0.0887	0.4327	3.8222	0.0503	0.9100
	0.7500	0.5505	0.1836	0.5982	3.5915	0.0396	0.8517
	1.0000	0.1969	0.2836	0.7136	3.0808	0.0137	0.6937
	1.2500	0.0	0.6041	0.3687	2.7000	0.0	0.5277
	1.5000	0.0	0.2022	-0.1010	2.7000	0.0	0.5277
	1.7500	0.0	0.1250	-0.1445	2.7000	0.0	0.5277
	2.0000	0.0	0.0672	0.0310	2.7000	0.0	0.5277
(2)	0.0	1.0869	0.0	0.3800	4.2103	0.0722	1.0038
	0.2500	1.0869	0.0	0.3800	4.2103	0.0722	1.0038
	0.5000	0.6684	0.0649	0.4247	3.7487	0.0464	0.8914
	0.7500	0.5433	0.2089	0.5837	3.5837	0.0390	0.8493
	1.0000	0.1233	0.2851	0.6878	2.9268	0.0100	0.6429
	1.2500	0.0	0.5798	0.3784	2.7000	0.0	0.5277
	1.5000	0.0	0.2384	-0.1266	2.7000	0.0	0.5277
	1.7500	0.0	0.0790	-0.1272	2.7000	0.0	0.5277
	2.0000	0.0	0.7397	0.1453	2.7000	0.0	0.5277

Table 16. Flow properties in Interior Region at t = 1.11 μsec after impact of a 2.5-cm-diameter projectile at 0.76 cm/μsec on an aluminum half space based on (1) Double Interpolation and (2) Orthogonal Polynomials Least Square Surface Fit with Double Interpolation after t = 1.1 μsec at Z = 0.25 cm.

Z = 0.25cm	r	P	U	V	ρ	e	a
(1)	0.0	1.0869	0.0	0.3800	4.2103	0.0722	1.0038
	0.2500	0.9839	0.0272	0.3851	4.1059	0.0659	0.9791
	0.5000	0.7084	0.0902	0.3852	3.7971	0.0489	0.9037
	0.7500	0.5403	0.1560	0.4163	3.5790	0.0388	0.8481
	1.0000	0.4323	0.2422	0.5445	3.4320	0.0315	0.8073
	1.2500	0.2618	0.3301	0.2324	3.1521	0.0215	0.7274
	1.5000	0.1760	0.2720	0.0015	3.0472	0.0120	0.6809
	1.7500	0.1315	0.1172	-0.0253	3.0716	0,0025	0.6606
	2.0000	0.0	0.0	0.0	2.7000	0.0	0.5277
(2)	0.0	1.0869	0.0	0.3800	4.2103	0.0722	1.0038
	0.2500	0.9982	0.0263	0.4170	4.1207	0.0668	0.9826
	0.5000	0.6946	0.0922	0.3700	3.7806	0.0480	0.8995
	0.7500	0.5285	0.1555	0.4120	3.5626	0.0381	0.8439
	1.0000	0.4507	0.2414	0.5557	3.4600	0.0325	0.8148
	1.2500	0.2818	0.3320	0.2294	3.1875	0.0226	0.7380
	1.5000	0.1784	0.2771	0.0039	3.0516	0.0122	0.6825
	1.7500	0.1331	0.1192	-0.0361	3.0754	0.0025	0.6618
	2.0000	0.0	0.0	0.0	2.7000	0.0	0.5277

Table 17. Flow properties in Interior Region at t = 1.11 μsec after impact of a 2.5-cm-diameter projectile at 0.76 cm/μsec on an aluminum half space based on (1) Double Interpolation and (2) Orthogonal Polynomials Least Square Surface Fit with Double Interpolation after t = 1.1 μsec at Z = 2.50 cm.

Z = 0.50 cm	r	P	U	y	ρ	e	a
(1)	0.0	1.0869	0.0	0.3800	4.2103	0.0722	1.0038
	0.2500	1.0058	0.0224	0.3776	4.1284	0.0672	0.9845
	0.5000	0.7873	0.0644	0.3560	3.8904	0.0538	0.9270
	0.7500	0.6405	0.1002	0.3370	3.7108	0.0450	0.8823
	1.0000	0.5742	0.1362	0.3663	3.6236	0.0410	0.8600
	1.2500	0.3283	0.1942	0.1842	3.2750	0.0246	0.7618
	1.5000	0.2347	0.1758	0.0773	3.2145	0.0110	0.7219
	1.7500	0.1343	0.0741	0.0354	3.0770	0.0026	0.6626
	2.0000	0.0	0.0	0.0	2.7000	0.0	0.5277
(2)	0.0	1.0869	0.0	0.3800	4.2103	0.0722	1.0038
	0.2500	1.0122	0.0136	0.3928	4.1350	0.0676	0.9860
	0.5000	0.7791	0.0687	0.3477	3.8810	0.0533	0.9246
	0.7500	0.6353	0.0971	0.3368	3.7042	0.0447	0.8806
	1.0000	0.5763	0.1374	0.3665	3.6261	0.0412	0.8607
	1.2500	0.3191	0.1977	0.1809	3.2574	0.0243	0.7572
	1.5000	0.2371	0.1742	0.0802	3.2194	0.0111	0.7233
	1.7500	0.1428	0.0846	0.0336	3.0931	0.0028	0.6679
	2.0000	0.0	0.0	0.0	2.7000	0.0	0.5277

Table 18. Flow properties in Interior Region at t = 1.1 μsec after impact of a 2.5-cm-diameter projectile at 0.76 cm/μsec on an aluminum half space based on (1) Double Interpolation and (2) Orthogonal Polynomials Least Square Surface Fit with Double Interpolation after t = 1.1 μsec at Z = 0.75 cm.

Z = 0.75 cm	r	P	U	V	ρ	e	a
(1)	0.0	1.0869	0.0	0.3800	4.2103	0.0722	1.0038
	0.2500	1.0575	0.0069	0.3782	4.1810	0.0704	0.9969
	0.5000	0.8898	0.0374	0.3551	4.0054	0.0601	0.9550
	0.7500	0.7535	0.0622	0.3208	3.8501	0.0518	0.9171
	1.0000	0.6515	0.0859	0.3099	3.7372	0.0442	0.8867
	1.2500	0.4555	0.1199	0.2073	3.5178	0.0276	0.8208
	1.5000	0.3195	0.1068	0,1168	3.4168	0.0112	0.7720
	1.7500	0.0	0.0	0.0	2.7000	0.0	0.5277
	2.0000	0.0	0.0	0.0	2.7000	0.0	0.5277
(2)	0.0	1.0869	0.0	0.3800	4.2103	0.0722	1.0038
	0.2500	0.9930	-0.0229	0.3930	4.1156	0.0664	0.9814
	0.5000	0.8864	0.0404	0.3514	4.1108	0.0599	0.9541
	0.7500	0.7504	0.0603	0.3195	3.8462	0.0517	0.9162
	1.0000	0.6505	0.0868	0.3100	3.7355	0.0442	0.8864
	1.2500	0.4552	0.1210	0.2011	3.5168	0.0277	0.8263
	1.5000	0.3313	0.1054	0.1197	3.4382	0.0116	0.7778
	1.7500	0.0	0.0	0.0	2.7000	0.0	0.5277
	2.0000	0.0	0.0	0.0	2.7000	0.0	0.5277

Table 19. Flow properties in Interior Region at t = 1.1 μsec after impact of a 2.5-cm-diameter projectile at 0.76 cm/μsec on an aluminum half space based on (1) Double Interpolation and (2) Orthogonal Polynomials Least Square Surface Fit with Double Interpolation after t = 1.1 μsec at Z = 1.00 cm.

Z = 1.00 cm	r	P	U	V	ρ	e	a
(1)	0.0	1.0869	0.0	0.3800	4.2103	0.0722	1.0038
	0.2500	1.0869	0.0	0.3800	4.2103	0.0722	1.0038
	0.5000	1.0119	0.0131	0.3673	4.1346	0.0676	0.9860
	0.7500	0.8990	0.0297	0.3359	4.0258	0.0593	0.9581
	1.0000	0.7910	0.0414	0.3143	3.9271	0.0499	0.9302
	1.2500	0.6043	0.0512	0.2428	3.7526	0.0330	0.8773
	1.5000	0.0	0.0	0.0	2.7000	0.0	0.5277
	1.7500	0.0	0.0	0.0	2.7000	0.0	0.5277
	2.0000	0.0	0.0	0.0	2.7000	0.0	0.5277
(2)	0.0	1.0869	0.0	0.3800	4.2103	0.0722	1.0038
	0.2500	1.0869	0.0	0.3800	4.2103	0.0722	1.0038
	0.5000	1.0110	0.0122	0.3680	4.1335	0.0676	0.9857
	0.7500	0.8941	0.0332	0.3364	4.0203	0.0590	0.9568
	1.0000	0.7734	0.0489	0.3178	3.9067	0.0488	0.9252
	1.2500	0.6046	0.0551	0.2587	3.7552	0.0328	0.8776
	1.5000	0.0	0.0	0.0	2.7000	0.0	0.5277
	1.7500	0.0	0.0	0.0	2.7000	0.0	0.5277
	2.0000	0.0	0.0	0.0	2.7000	0.0	0.5277

```
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPS
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LEN
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(
     115,4),RPART(15,2)
C
C
      COMMON ZO,RO,PO,UO,VO,LO,MO,RHOO,EO,AO,UBARO,VBARO
C
      COMMON NP,NT,NR,NI,NDEL,ISUB
C
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H
      COMMON DIRCOS
      COMMON TIME
      COMMON IRARF
      COMMON KSTOP
      COMMON TPSI
      COMMON KKK
C
C
      REAL LO,MO,LENGTH,MU,KO
      DOUBLE PRECISION PHI(20,20,6)
      KR=7
      EPS=.0000001
      KSTOP=0
1     FORMAT (1H1)
      CALL DVCHK(KEY)
      KICK=0
      IF (KEY.EQ.1) GO TO 9980
C
      DO 2 K=1,6
      DO 2 J=1,20
      DO 2 I=1,20
      XMESH(I,J,K)=0.
      XMESH2(I,J,K)=0.
```

```
2     CONTINUE                                                          36
C     KRW=0                                                             37
      NUZON=0                                                           38
C                                                                       39
C                                                                       40
      WRITE (3,4)                                                       41
C                                                                       42
C         DATA INPUT SECTION                                            43
C                                                                       44
4     FORMAT (52H1HYPERVELOCITY IMPACT METHOD OF CHARACTERISTICS CODE///  45
     1)                                                                 46
C       ID AND FX. PT. CONSTANTS                                        47
C                                                                       48
      READ (1,8) CASEID,ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,NDEL   49
8     FORMAT (13A6,A2/9I3)                                              50
      IRARF=ITI2                                                        51
C                                                                       52
C       FL. PT. CONSTANTS                                               53
C                                                                       54
      READ (1,15) EPS1,EPS2,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5 55
     1,EPI6,EPI7                                                        56
      READ (1,14) NP,NT,NR                                              57
14    FORMAT (3I3)                                                      58
      READ (1,15) APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,REF 59
     1L                                                                 60
      READ (1,15) ZMIN,ZMAX,RMIN,RMAX,GR,GZ,DELTA,VP,LENGTH,RADIUS,HSTAR 61
15    FORMAT (6E12.8)                                                   62
      READ (1,15) RST                                                   63
      IF (RST.GT.0.) GO TO 16                                           64
      REWIND 5
C     DO 1529 JTP=1,200                                                 66
      READ (5)TIME                                                      67
C     IF(ABS(RST+TIME).LT..001) GO TO 1530                              68
C     READ(5)BLOB
C1529 CONTINUE                                                          70
1530  CONTINUE                                                          71
```

```
C      KRW=1                                                                  72
       READ (5)(((XMESH(I,J,K),I=1,20),J=1,20),K=1,6),(Z(I),I=1,20),(R(I)     73
      1,I=1,20),((SURF(I,J),I=1,15),J=1,8),(((TAB(I,J,K),I=1,15),J=1,14),     74
      1K=1,2),((RARF(I,J),I=1,15),J=1,11),TIME,ZMIN,ZMAX,RMIN,RMAX,GR,GZ,     75
      1AR                                                                    76
       DO 1500 J=1,20                                                        77
       DO 1500 I=1,20                                                        78
       DO 1500 K=1,6                                                         79
1500   XMESH2(I,J,K)=XMESH(I,J,K)                                            80
       DO 1501 I=1,15                                                        81
       DO 1501 J=1,8                                                         82
1501   SURF2(I,J)=SURF(I,J)                                                  83
       WRITE (3,145) TIME                                                    84
       CALL SOUT                                                             85
       CALL PRINT(XMESH2,Z,R,1)                                              86
       KREFL=0                                                               87
16     CONTINUE                                                              88
       WRITE (3,10) CASEID,ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,NDEL      89
10     FORMAT (1X13A6,A2//17H SHOCK ITERATIONS6X,4I4//20H INTERIOR ITERAT     90
      1IONS3X,4I4//7H NDEL =,I4///)                                          91
       WRITE (3,18) EPS1,EPS2,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI    92
      15,EPI6,EPI7,ZMIN,ZMAX,RMIN,RMAX,DELTA,VP,LENGTH,RADIUS,APR,BPR,BIG    93
      1APR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,REFL                          94
18     FORMAT (///38H ERROR CRITERIA FOR SHOCK COMPUTATIONS//5X8HDELTA Z1     95
      18X8HDELTA R18X9HDELTA RHO7X7HDELTA E9X7HDELTA P9X7HDELTA U/6E16.6/    96
      1//41H ERROR CRITERIA FOR INTERIOR COMPUTATIONS//5X8HDELTA ZI8X8HDE     97
      1LTA RI8X7HDELTA P9X7HDELTA U9X7HDELTA V9X9HDELTA RHO7X7HDELTA E/7E     98
      116.6///5X4HZMIN12X4HZMAX12X4HRMIN12X4HRMAX12X5HDELTA11X2HVP14X6HLE   99
      1NGTH10X6HRADIUS/8E16.6///5X2HA'14X2HB'14X6HBIG A'10X6HBIG B'10X2HE   100
      1*14X5HALPHA11X4HBETA12X4HRHO*/8E16.6//5X3HE'S13X4HREFL/2E16.6///)    101
       IF (RST.LT.0.) GO TO 140                                              102
C                                                                            103
C          STORE RHO* IN ALL XMESH                                           104
C                                                                            105
       DO 22 J=1,20                                                          106
       DO 22 I=1,20                                                          107
```

```
22    XMESH(I,J,4)=RHOSTR                                                    108
40    FORMAT (5E12.8)                                                        109
C                                                                            110
C       PROJECTILE SHOCK                                                     111
C                                                                            112
      CALL EQOS1(PRHO,PPP,PVV,PEE,TEE,TRHO,KICK)                             113
      IF (KICK.EQ.2200) GO TO 9980                                           114
      DO 230 N=1,NP                                                          115
      TAB(N,1,1)=(PRHO*PVV-RHOSTR*VP)*HSTAR/(PRHO-RHOSTR)                    116
      EE=N-1                                                                 117
      FNP=NP                                                                 118
      TAB(N,2,1)=RMIN+EE*(RADIUS-RMIN)/FNP                                   119
      TAB(N,3,1)=PPP                                                         120
      TAB(N,4,1)=0.0                                                         121
      TAB(N,5,1)=PVV                                                         122
      TAB(N,6,1)=PRHO                                                        123
      TAB(N,7,1)=PEE                                                         124
      TAB(N,9,1)=0.0                                                         125
      TAB(N,10,1)=1.0                                                        126
230   CONTINUE                                                               127
C                                                                            128
C       TARGET SHOCK                                                         129
C                                                                            130
      M=0                                                                    131
      DO 240 N=1,NT                                                          132
      EE=N-1                                                                 133
      FNT=NT-4                                                               134
      TAB(N,2,2)=RMIN+EE*(RADIUS-RMIN)/FNT                                   135
      TAB(N,7,2)=TEE                                                         136
      TAB(N,6,2)=TRHO                                                        137
      TAB(N,3,2)=PPP                                                         138
      IF (TAB(N,2,2).GT.RADIUS) GO TO 250                                    139
      TAB(N,1,2)=TRHO*PVV*HSTAR/(TRHO-RHOSTR)                                140
      TAB(N,4,2)=0.0                                                         141
      TAB(N,5,2)=PVV                                                         142
      TAB(N,9,2)=0.0                                                         143
```

```
      TAB(N,10,2)=1.0                                                   144
      GO TO 240                                                         145
250   EF=M                                                              146
      TAB(N,9,2)=SIN(.5236+EF*.2618)                                    147
      TAB(N,10,2)=COS(.5236+EF*.2618)                                   148
      TAB(N,4,2)=PVV*TAB(N,9,2)                                         149
      TAB(N,5,2)=PVV*TAB(N,10,2)                                        150
      TAB(N,1,2)=TAB(1,1,2)*TAB(N,10,2)                                 151
      TAB(N,2,2)=RADIUS+TAB(1,1,2)*TAB(N,9,2)                           152
      M=M+1                                                             153
240   CONTINUE                                                          154
C                                                                       155
C       RAREFACTION                                                     156
C                                                                       157
      CALL EQOS2(PPP,PRHO,PEE)                                          158
      EE=NR-1                                                           159
      ADEL=(TAB(1,1,2)-TAB(1,1,1))/EE                                   160
      RARF(1,1)=TAB(1,1,2)                                              161
      DO 205 N=2,NR                                                     162
      RARF(N,1)=RARF(N-1,1)-ADEL                                        163
205   CONTINUE                                                          164
      DO 210 N=1,NR                                                     165
      RARF(N,10)=(RARF(N,1)/HSTAR-.5*VP)/AR                             166
      RARF(N,9)=-SQRT(1.-RARF(N,10)**2)                                 167
      RARF(N,2)=RADIUS+HSTAR*AR*RARF(N,9)                               168
210   CONTINUE                                                          169
      DO 220 N=1,NR                                                     170
      RARF(N,3)=TAB(1,3,1)                                              171
      RARF(N,4)=0.                                                      172
      RARF(N,5)=TAB(1,5,1)                                              173
      RARF(N,6)=TAB(1,6,1)                                              174
      RARF(N,7)=TAB(1,7,1)                                              175
220   CONTINUE                                                          176
C       REGION INTERIOR TO SHOCKS                                       177
      I=-ZMIN/GZ+1.2                                                    178
      J=0                                                               179
```

```
260   J=J+1                                                          180
      XMESH(I,J,1)=PPP                                               181
      XMESH(I,J,3)=PVV                                               182
      XMESH(I,J,4)=PRHO                                              183
      XMESH(I,J,5)=PEE                                               184
      EE=(J-1)                                                       185
      IF ((EE*GR-RADIUS).LT.-EPS) GO TO 260                          186
      XMESH(I,J,1)=0.                                                187
      XMESH(I,J,4)=RHOSTR                                            188
      XMESH(I,J,5)=0.                                                189
C                                                                    190
C       FREE SURFACE                                                 191
C                                                                    192
      DO 50 I=1,NP                                                   193
      SURF(I,1)=-LENGTH+VP*HSTAR                                     194
      SURF(I,2)=TAB(I,2,1)                                           195
      SURF(I,3)=0.                                                   196
      SURF(I,4)=0.                                                   197
      SURF(I,5)=VP                                                   198
      SURF(I,6)=RHOSTR                                               199
      SURF(I,7)=0.                                                   200
      SURF(I,8)=SQRT(BIGAPR/RHOSTR)                                  201
50    CONTINUE                                                       202
      DO 51 I=1,NP                                                   203
      DO 51 J=1,8                                                    204
51    SURF2(I,J)=SURF(I,J)                                           205
      IF (NUZON.EQ.0) GO TO 5001                                     206
5000  GR=GR*2.                                                       207
      GZ=GZ*2.                                                       208
      ZMAX=ZMAX*2.-ZMIN                                              209
      RMAX=RMAX*2.                                                   210
      NUZON=1                                                        211
      WRITE (3,5003)                                                 212
5003  FORMAT (7H REZONE///)                                          213
5001  CONTINUE                                                       214
C                                                                    215
```

```
C
      DO 55 I=1,20                                                      216
      EE=I-1                                                            217
      Z(I)=ZMIN+EE*GZ                                                   218
      R(I)=RMIN+EE*GR                                                   219
55    CONTINUE                                                          220
      IF (NUZON.EQ.0) GO TO 5101                                        221
      DO 5100 I=1,10                                                    222
      DO 5100 J=1,10                                                    223
      DO 5100 K=1,6                                                     224
      L=2*I-1                                                           225
      M=2*J-1                                                           226
      XMESH(I,J,K)=XMESH(L,M,K)                                         227
5100  CONTINUE                                                          228
      GO TO 157                                                         229
5101  CONTINUE                                                          230
C                                                                       231
C          COMPUTE A FOR 2 SHOCKS AND MESH                              232
C                                                                       233
      DO 86 K=1,3                                                       234
      GO TO (57,59,61),K                                                235
57    NN=NP                                                             236
      JJ=1                                                              237
      GO TO 63                                                          238
59    NN=NT                                                             239
      JJ=1                                                              240
      GO TO 63                                                          241
61    NN=20                                                             242
      JJ=20                                                             243
63    DO 84 N=1,NN                                                      244
      DO 82 J=1,JJ                                                      245
      GO TO (65,65,68),K                                                246
65    P=TAB(N,3,K)                                                      247
      RHO=TAB(N,6,K)                                                    248
      E=TAB(N,7,K)                                                      249
      GO TO 70                                                          250
                                                                        251
```

```
68    P=XMESH(J,N,1)                                                          252
      RHO=XMESH(J,N,4)                                                        253
      E=XMESH(J,N,5)                                                          254
70    CONTINUE                                                                255
      CALL EQOS3(RHO,AA,E,P)                                                  256
      GO TO (76,76,78),K                                                      257
76    TAB(N,8,K)=AA                                                           258
      GO TO 82                                                                259
78    XMESH(J,N,6)=AA                                                         260
82    CONTINUE                                                                261
84    CONTINUE                                                                262
86    CONTINUE                                                                263
      KICK=86                                                                 264
      CALL DVCHK(KQ)                                                          265
      IF (KQ.EQ.1) GO TO 9980                                                 266
C                                                                             267
C         STORE A FOR RAREFACTION                                             268
C                                                                             269
      DO 90 I=1,NR                                                            270
90    RARF(I,8)=AR                                                            271
C                                                                             272
C         COMPLETE SHOCK TABLES                                               273
C                                                                             274
      DO 99 K=1,2                                                             275
      GO TO (92,94),K                                                         276
92    NN=NP                                                                   277
      GO TO 95                                                                278
94    NN=NT                                                                   279
      US=0.                                                                   280
95    DO 97 N=1,NN                                                            281
      GO TO (93,96),K                                                         282
93    CONTINUE                                                                283
      US=VP*TAB(N,10,1)                                                       284
96    CONTINUE                                                                285
      TAB(N,11,K)=TAB(N,9,K)*TAB(N,4,K)+TAB(N,10,K)*TAB(N,5,K)                286
      TAB(N,12,K)=TAB(N,9,K)*TAB(N,5,K)-TAB(N,10,K)*TAB(N,4,K)                287
```

```
      TAB(N,13,K)=((TAB(N,6,K)*ABS(TAB(N,11,K)))/(TAB(N,6,K)-RHOSTR)-US)
     1*(-1.)**K
      TAB(N,14,K)=1.
97    CONTINUE
99    CONTINUE
C
C
C         STORE ALL XMESH IN XMESH2
C
130   DO 135 K=1,6
      DO 135 J=1,20
      DO 135 I=1,20
      XMESH2(I,J,K)=XMESH(I,J,K)
135   CONTINUE
      TIME=HSTAR
      WRITE (3,145) TIME
      CALL SOUT
C     CALL PRINT(XMESH2,Z,R,1)
      CALL PRINT(XMESH2,Z,R,1,PHI,NMAX,MMAX)
C
      KREFL=0
139   IF (KREFL.NE.0) GO TO 143
C         ENTRY FOR TIME STEP
C
140   READ (1,142) H
142   FORMAT (E12.8)
143   CONTINUE
      TIME=TIME+H
      WRITE (3,145) TIME
145   FORMAT (1H1///6H TIME=,E15.8///)
      WRITE (3,999) KR
999   FORMAT (5X,4H KR=,I5)
C
C
C         ADVANCE SHOCK POINTS
C
```

```
      DO 159 N=1,NP                                                          323
      IF (TAB(N,14,1).LT.0.) GO TO 156                                       324
      IF ((TAB(N,1,1)-SURF(N,1)).GT.EPS) GO TO 154                           325
156   TAB2(N,1,1)=TAB(N,1,1)                                                 326
      TAB2(N,2,1)=TAB(N,2,1)                                                 327
      TAB(N,14,1)=-1.                                                        328
      GO TO 159                                                              329
154   TAB2(N,1,1)=TAB(N,1,1)+TAB(N,13,1)*H*TAB(N,10,1)-VP*TAB(N,9,1)*TAB     330
     1(N,10,1)*H                                                             331
150   TAB2(N,2,1)=TAB(N,2,1)+TAB(N,13,1)*H*TAB(N,9,1)-VP*TAB(N,9,1)**2*H     332
159   CONTINUE                                                               333
      DO 155 N=1,NT                                                          334
      TAB2(N,1,2)=TAB(N,1,2)+TAB(N,13,2)*H*TAB(N,10,2)                       335
155   TAB2(N,2,2)=TAB(N,2,2)+TAB(N,13,2)*H*TAB(N,9,2)                        336
      DO 158 M=1,NT                                                          337
      IF (TAB2(M,1,2).GT.ZMAX) GO TO 5000                                    338
      IF (TAB2(M,2,2).GT.RMAX) GO TO 5000                                    339
158   CONTINUE                                                               340
157   NUZON=0                                                                341
C         ADVANCE RAREFACTION                                                342
C                                                                            343
      IF (RARF(1,2).LT.0.)IRARF=1                                            344
      IF (IRARF.EQ.1) GO TO 516                                              345
      ENR=NR-1                                                               346
      ADEL=(TAB2(1,1,2)-TAB2(1,1,1))/ENR                                     347
      RARF2(1,1)=TAB2(1,1,2)                                                 348
      DO 510 N=2,NR                                                          349
      RARF2(N,1)=RARF2(N-1,1)-ADEL                                           350
510   CONTINUE                                                               351
      DO 515 N=1,NR                                                          352
      RARF2(N,3)=(RARF2(N,1)/TIME-.5*VP)/AR                                  353
      RARF2(N,4)=-SQRT(1.-RARF2(N,3)**2)                                     354
      RARF2(N,2)=RADIUS+TIME*AR*RARF2(N,4)                                   355
515   CONTINUE                                                               356
516   CONTINUE                                                               357
      CALL SHOCK                                                             358
```

```
C                                                                       359
C                                                                       360
      IF (ITS3.EQ.1) GO TO 569                                          361
C         SHOCK COMPUTATIONS COMPLETED                                  362
C         COMPUTE PARTICLE CURVES                                       363
C                                                                       364
      TMP=.5*VP*H                                                       365
      DO 520 N=1,NP                                                     366
      SPART(N,1,1)=TAB(N,1,1)+TAB(N,5,1)*H                              367
520   SPART(N,2,1)=TAB(N,2,1)+TAB(N,4,1)*H                              368
      DO 525 N=1,NT                                                     369
      SPART(N,1,2)=TAB(N,1,2)+TAB(N,5,2)*H                              370
525   SPART(N,2,2)=TAB(N,2,2)+TAB(N,4,2)*H                              371
      IF (IRARF.EQ.1) GO TO 531                                         372
      DO 530 N=1,NR                                                     373
      RPART(N,1)=RARF(N,1)+TMP                                          374
530   RPART(N,2)=RARF(N,2)                                              375
531   CONTINUE                                                          376
C                                                                       377
C                                                                       378
568   CALL SOUT2                                                        379
569   CONTINUE                                                          380
C       ADVANCE PROJECTILE REAR SURFACE                                 381
      DO 5300 I=1,NP                                                    382
      DO 5300 J=1,8                                                     383
5300  SURF2(I,J)=SURF(I,J)                                              384
      KICK=568                                                          385
      CALL DVCHK(KQ)                                                    386
      IF (KQ.EQ.1) GO TO 9980                                           387
C                                                                       388
C         START INTERIOR REGION COMPUTATIONS                            389
C                                                                       390
C     CALL INTER
      CALL INTER(PHI,NMAX,MMAX)
C                                                                       392
C         INTERIOR REGION COMPUTATIONS COMPLETED                        393
```

```
C                                                                       394
C     CALL PRINT(XMESH2,Z,R,2)
      CALL PRINT(XMESH2,Z,R,2,PHI,NMAX,MMAX)
570   CONTINUE                                                          396
C                                                                       397
C          INITIALIZE FOR NEXT TIME STEP                                398
C                                                                       399
      DO 920 K=1,6                                                      400
      DO 920 J=1,20                                                     401
      DO 920 I=1,20                                                     402
      XMESH(I,J,K)=XMESH2(I,J,K)                                        403
920   CONTINUE                                                          404
      DO 930 J=1,13                                                     405
      DO 930 I=1,NP                                                     406
      TAB(I,J,1)=TAB2(I,J,1)                                            407
930   CONTINUE                                                          408
      DO 940 J=1,13                                                     409
      DO 940 I=1,NT                                                     410
      TAB(I,J,2)=TAB2(I,J,2)                                            411
940   CONTINUE                                                          412
      IF (IRARF.EQ.1) GO TO 951                                         413
      DO 950 I=1,NR                                                     414
      RARF(I,1)=RARF2(I,1)                                              415
      RARF(I,2)=RARF2(I,2)                                              416
950   CONTINUE                                                          417
951   CONTINUE                                                          418
      DO 960 I=1,NP                                                     419
      DO 959 J=1,5                                                      420
      SURF(I,J)=SURF2(I,J)                                              421
959   CONTINUE                                                          422
960   CONTINUE                                                          423
777   IF (KREFL.EQ.0) GO TO 980                                         442
      KREFL=0                                                           443
      H=H1                                                              444
      GO TO 143                                                         445
980   CONTINUE                                                          446
```

```
      CALL DVCHK(KICK)                                                  447
      IF (KICK.EQ.2) GO TO 140                                          448
      WRITE (3,970)                                                     449
970   FORMAT (28HODIVIDE CHECK AT END OF CASE/1H1)                      450
      CALL EXIT                                                         451
C                                                                       452
C         DIVIDE CHECK                                                  453
C                                                                       454
9980  WRITE (3,9985) KICK                                               455
9985  FORMAT (32HODIVIDE CHECK NEAR STATEMENT NO.,I5/1H1)               456
      RETURN                                                            457
      DEBUG SUBCHK
      END                                                               458
      SUBROUTINE DBLTRP(ZX,RX,ANS)                                        1
C                                                                         2
C         1ST ORDER DOUBLE INTERPOLATION THAT CONSIDERS                   3
C         LINES OF DISCONTINUITY IF IN CONSIDERED REGION                  4
C                                                                         5
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPS  6
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LEN  7
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS         8
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2   9
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(  10
     115,4),RPART(15,2)                                                   11
C                                                                         12
C                                                                         13
      COMMON ZO,RO,PO,UO,VO,LO,MO,RHOO,EO,AO,UBARO,VBARO                  14
C                                                                         15
      COMMON NP,NT,NR,NI,NDEL,ISUB                                        16
C                                                                         17
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                     18
      COMMON DIRCOS                                                       19
      COMMON TIME                                                         20
      COMMON IRARF                                                        21
      COMMON KSTOP                                                        22
      COMMON TPSI                                                         23
```

```
      COMMON KKK                                                        24
      REAL LO,MO,LENGTH,MU,KO                                           25
C                                                                       26
      DIMENSION ANS(6),ANS1(2,8),ANS2(2,8),ZI(4),RI(4),IK(4)            27
      CALL DVCHK(KEY)                                                   28
      IF (KEY.EQ.2) GO TO 4                                             29
      NO=0                                                              30
      GO TO 940                                                         31
C                                                                       32
C         FIND SUBSCRIPTS FOR GRID                                      33
C                                                                       34
4     I1=(ZX-ZMIN)/GZ+1.000001                                          35
      I2=I1+1                                                           36
      J1=(RX-RMIN)/GR+1.000001                                          37
      J2=J1+1                                                           38
      NN=NP                                                             39
      IF (ITS3.EQ.1) GO TO 3                                            40
      DO 1 K=1,2                                                        41
      IF (K.EQ.2)NN=NT                                                  42
      DO 1 I=1,NN                                                       43
      ALF=SQRT((TAB(I,1,K)-ZX)**2+(TAB(I,2,K)-RX)**2)                   44
      IF (ALF.GT.EPS1) GO TO 1                                          45
      ANS(1)=TAB(I,3,K)                                                 46
      ANS(2)=TAB(I,4,K)                                                 47
      ANS(3)=TAB(I,5,K)                                                 48
      ANS(4)=TAB(I,6,K)                                                 49
      ANS(5)=TAB(I,7,K)                                                 50
      ANS(6)=TAB(I,8,K)                                                 51
      RETURN                                                            52
1     CONTINUE                                                          53
      IF (IRARF.EQ.1) GO TO 3                                           54
      DO 2 I=1,NR                                                       55
      ALF=SQRT((RARF(I,1)-ZX)**2+(RARF(I,2)-RX)**2)                     56
      IF (ALF.GT.EPI1) GO TO 2                                          57
      ANS(1)=RARF(I,3)                                                  58
      ANS(2)=RARF(I,4)                                                  59
```

```
      ANS(3)=RARF(I,5)                                                  60
      ANS(4)=RARF(I,6)                                                  61
      ANS(5)=RARF(I,7)                                                  62
      ANS(6)=RARF(I,8)                                                  63
      RETURN                                                            64
2     CONTINUE                                                          65
3     CONTINUE                                                          66
C                                                                       67
C                                                                       68
      ZXX=ZX+.01                                                        69
      RXX=RX+.01                                                        70
C                                                                       71
C         I LOOP FOR UPPER AND LOWER Z GRID LINES                       72
C                                                                       73
      DO 800 I=1,2                                                      74
      IF (ITS3.EQ.1) GO TO 14                                           75
      IF (I.EQ.2) GO TO 8                                               76
      II=I1                                                             77
      GO TO 12                                                          78
8     II=I2                                                             79
12    M=COMP(ZX,RX,Z(II),R(J1))                                         80
      IF (M.EQ.1) GO TO 13                                              81
      MCOM=1                                                            82
      GO TO 20                                                          83
13    M=COMP(ZX,RX,Z(II),R(J2))                                         84
      IF (M.EQ.1) GO TO 14                                              85
      MCOM=2                                                            86
      GO TO 20                                                          87
C                                                                       88
C         GET 6 VALUES ON GRID LINES                                    89
C                                                                       90
14    DO 15 K=1,6                                                       91
      ANS1(I,K+2)=XMESH(II,J1,K)+(XMESH(II,J2,K)-XMESH(II,J1,K))*(RX-R(J  92
     11))/(R(J2)-R(J1))                                                 93
15    CONTINUE                                                          94
      CALL DVCHK(NO)                                                    95
```

```
      IF (NO.EQ.2) GO TO 17                                                  96
      NO=15                                                                  97
      GO TO 940                                                              98
17    ANS1(I,1)=Z(II)                                                        99
      ANS1(I,2)=RX                                                          100
      GO TO 800                                                             101
C                                                                           102
C                                                                           103
20    ZZ=Z(II)                                                              104
      RR=RX                                                                 105
      M=COMP(ZX,RX,ZZ,RR)                                                   106
      IF (M.EQ.1) GO TO 300                                                 107
C                                                                           108
C                                                                           109
      DO 25 K=1,2                                                           110
      KATCH=0                                                               111
      GO TO (21,22),K                                                       112
21    NN=NP                                                                 113
      GO TO 23                                                              114
22    NN=NT                                                                 115
23    JJJ=NN-1                                                              116
      DO 24 M=1,JJJ                                                         117
      IF (M.EQ.JJJ) GO TO 210                                               118
      IF (RX.GT.TAB(M+1,2,K).OR.RX.LT.TAB(M,2,K)) GO TO 24                  119
210   ZI(K)=TAB(M+1,1,K)+(TAB(M,1,K)-TAB(M+1,1,K))*(RX-TAB(M+1,2,K))/(TA    120
     1B(M,2,K)-TAB(M+1,2,K))                                                121
      NO=210                                                                122
      CALL DVCHK(KQ)                                                        123
      IF (KQ.EQ.1) GO TO 940                                                124
      IF (I.EQ.2) GO TO 211                                                 125
      IF (KATCH.EQ.1) GO TO 212                                             126
      KATCH=1                                                               127
      GO TO 213                                                             128
212   IF ((ZX-ZI(K)).GT.(ZX-ZM)) GO TO 24                                   129
      IF (ZI(K).GT.ZX) GO TO 24                                             130
213   ZM=ZI(K)                                                              131
```

```
      IF (K.EQ.2) GO TO 215                                                  132
      NPS=M                                                                  133
      GO TO 24                                                               134
215   NTS=M                                                                  135
      GO TO 24                                                               136
211   CONTINUE                                                               137
      IF (KATCH.EQ.1) GO TO 26                                               138
      KATCH=1                                                                139
      GO TO 213                                                              140
26    IF ((ZI(K)-ZX).GT.(ZM-ZX)) GO TO 24                                    141
      IF (ZI(K).LT.ZX) GO TO 24                                              142
      GO TO 213                                                              143
24    CONTINUE                                                               144
      ZI(K)=ZM                                                               145
      IF (KATCH.NE.0) GO TO 25                                               146
      ZI(K)=ZMAX+1.                                                          147
25    CONTINUE                                                               148
      IF (IRARF.EQ.1)ZI(3)=ZMAX+1.                                           149
      IF (IRARF.EQ.1) GO TO 2504                                             150
      KATCH=0                                                                151
      JJJ=NR-1                                                               152
      DO 27 M=1,JJJ                                                          153
      ZI(3)=RARF(M+1,1)+(RARF(M,1)-RARF(M+1,1))*(RX-RARF(M+1,2))/(RARF(M     154
     1,2)-RARF(M+1,2))                                                       155
      NO=25                                                                  156
      CALL DVCHK(KQ)                                                         157
      IF (KQ.EQ.1) GO TO 940                                                 158
      IF (ABS(ZI(3)-ZX).GT.1.E-5) GO TO 279                                  159
      DO 2799 LN=1,6                                                         160
      LNN=LN+2                                                               161
      ANS(LN)=RARF(1,LNN)                                                    162
2799  CONTINUE                                                               163
      GO TO 820                                                              164
279   CONTINUE                                                               165
      IF (I.EQ.2) GO TO 28                                                   166
      IF (ZI(3).GT.ZX) GO TO 27                                              167
```

```
      IF (KATCH.EQ.1) GO TO 280                                          168
      KATCH=1                                                            169
      GO TO 281                                                          170
280   IF ((ZX-ZI(3)).GT.(ZX-ZM)) GO TO 27                                171
281   ZM=ZI(3)                                                           172
      MR=M                                                               173
      GO TO 27                                                           174
28    IF (ZI(3).LT.ZX) GO TO 27                                          175
      IF (KATCH.EQ.1) GO TO 282                                          176
      KATCH=1                                                            177
      GO TO 281                                                          178
282   IF ((ZI(3)-ZX).GT.(ZM-ZX)) GO TO 27                                179
      GO TO 281                                                          180
27    CONTINUE                                                           181
      ZI(3)=ZM                                                           182
2504  CONTINUE                                                           183
      KATCH=0                                                            184
      K=4                                                                185
      JJJ=NP-1                                                           186
      DO 2700 M=1,JJJ                                                    187
      IF (M.EQ.JJJ) GO TO 2710                                           188
      IF (RX.GT.SURF(M+1,2).OR.RX.LT.SURF(M,2)) GO TO 2700               189
2710  ZI(4)=SURF(M+1,1)+(SURF(M,1)-SURF(M+1,1))*(RX-SURF(M+1,2))/(SURF(M 190
     1,2)-SURF(M+1,2))                                                   191
      IF (KQ.EQ.1) GO TO 940                                             192
      CALL DVCHK(KQ)                                                     193
      NO=2710                                                            194
      IF (I.EQ.2) GO TO 2711                                             195
      IF (KATCH.EQ.1) GO TO 2712                                         196
      KATCH=1                                                            197
      GO TO 2713                                                         198
2712  IF ((ZX-ZI(K)).GT.(ZX-ZM)) GO TO 2700                              199
2713  ZM=ZI(4)                                                           200
      MS=M                                                               201
      GO TO 2700                                                         202
2711  CONTINUE                                                           203
```

```
      IF (KATCH.EQ.1) GO TO 2726
      KATCH=1
      GO TO 2713
2726  IF ((ZI(K)-ZX).GT.(ZM-ZX)) GO TO 2700
      GO TO 2713
2700  CONTINUE
      ZI(4)=ZM
      IF (KATCH.NE.0) GO TO 2701
      ZI(4)=ZMAX+1.
2701  CONTINUE
      RI(1)=RX
      RI(2)=RX
      RI(3)=RX
      RI(4)=RX
C          FIND INTERSECTION TO USE
C
30    KEY=0
      IF (I.EQ.2) GO TO 50
C
C          UPPER GRID LINE
C
      DO 40 KK=1,4
      IF (Z(II).GT.ZI(KK)) GO TO 40
      IF (ABS(ZI(KK)-ZX).LT.1.E-5) GO TO 35
      IF (ZI(KK).GT.ZX) GO TO 40
      IF (KEY.EQ.0) GO TO 35
      IF (ZI(KK).LE.ZI(KEEP)) GO TO 40
      KEEP=KK
      GO TO 40
35    KEEP=KK
      KEY=1
40    CONTINUE
      GO TO 65
C
C          LOWER GRID LINE
C
```

```
50    DO 60 KK=1,4                                                          240
      IF (Z(II).LT.ZI(KK)) GO TO 60                                         241
      IF (ABS(ZI(KK)-ZX).LT.1.E-5) GO TO 55                                 242
      IF (ZI(KK).LT.ZX) GO TO 60                                            243
      IF (KEY.EQ.0) GO TO 55                                                244
      IF (ZI(KK).GE.ZI(KEEP)) GO TO 60                                      245
      KEEP=KK                                                               246
      GO TO 60                                                              247
55    KEEP=KK                                                               248
      KEY=1                                                                 249
60    CONTINUE                                                              250
C                                                                           251
65    IF (KEY.NE.0) GO TO 70                                                252
      WRITE (6,67) ZX,RX,I,(ZI(KEY),KEY=1,4),(RI(KEY),KEY=1,4)              253
67    FORMAT (34HOERROR NEAR STATEMENT 65 IN DBLTRP/1X3HZX=,E15.8,4X3HRX    254
     1=,E15.8,4X2HI=,I3/1X3HZI=,4E20.8/1X3HRI=,4E20.8/1H1)                  255
      XYZ=-2.                                                               256
      ZYX=SQRT(XYZ)                                                         257
      CALL EXIT                                                             258
C                                                                           259
C         FIND 6 VALUES ON SELECTED DISCONTINUITY                           260
C                                                                           261
70    IF (KEEP.EQ.3) GO TO 80                                               262
      IF (KEEP.EQ.4) GO TO 81                                               263
      IF (KEEP.EQ.2) GO TO 71                                               264
      N=NPS                                                                 265
      GO TO 72                                                              266
71    N=NTS                                                                 267
72    CONTINUE                                                              268
      ZY=ZI(KEEP)                                                           269
      RY=RX                                                                 270
      DO 75 K=3,8                                                           271
      ANS1(I,K)=TAB(N,K,KEEP)+(TAB(N+1,K,KEEP)-TAB(N,K,KEEP))*SQRT(((RY-    272
     1TAB(N,2,KEEP))**2+(ZY-TAB(N,1,KEEP))**2)/((TAB(N+1,2,KEEP)-TAB(N,2   273
     1,KEEP))**2+(TAB(N+1,1,KEEP)-TAB(N,1,KEEP))**2))                      274
      NO=75                                                                 275
```

```
      CALL DVCHK(KQ)                                                    276
      IF (KQ.EQ.1) GO TO 940                                            277
75    CONTINUE                                                          278
      GO TO 90                                                          279
80    N=MR                                                              280
      ZY=ZI(3)                                                          281
      RY=RX                                                             282
      DO 85 K=3,8                                                       283
      ANS1(I,K)=RARF(N,K)+(RARF(N+1,K)-RARF(N,K))*SQRT(((RY-RARF(N,2))** 284
     12+(ZY-RARF(N,1))**2)/((RARF(N+1,2)-RARF(N,2))**2+(RARF(N+1,1)-RARF 285
     1(N,1))**2))                                                       286
      NO=85                                                             287
      CALL DVCHK(KQ)                                                    288
      IF (KQ.EQ.1) GO TO 940                                            289
85    CONTINUE                                                          290
      GO TO 90                                                          291
81    N=MS                                                              292
      ZY=ZI(4)                                                          293
      RY=RX                                                             294
      DO 86 K=3,8                                                       295
      ANS1(I,K)=SURF(N,K)+(SURF(N+1,K)-SURF(N,K))*SQRT(((RY-SURF(N,2))** 296
     12+(ZY-SURF(N,1))**2)/((SURF(N+1,2)-SURF(N,2))**2+(SURF(N+1,1)-SURF 297
     1(N,1))**2))                                                       298
86    CONTINUE                                                          299
90    CALL DVCHK(NO)                                                    300
      IF (NO.EQ.2) GO TO 92                                             301
      NO=90                                                             302
      GO TO 940                                                         303
92    ANS1(I,1)=ZY                                                      304
      ANS1(I,2)=RY                                                      305
      GO TO 800                                                         306
C                                                                       307
C         FIND INTERSECTIONS OF 3 DISCONTINUITIES AND Z GRID LINE       308
C                                                                       309
300   CONTINUE                                                          310
      KATCHP=0                                                          311
```

```
      KATCHR=0                                                          312
      KATCHT=0                                                          313
      KATCHS=0                                                          314
      DO 310 K=1,2                                                      315
      KATCH=0                                                           316
      GO TO (303,301),K                                                 317
303   NN=NP                                                             318
      GO TO 302                                                         319
301   NN=NT                                                             320
302   JJJ=NN-1                                                          321
      DO 309 M=1,JJJ                                                    322
      IF ((TAB(M,1,K)-TAB(M+1,1,K)).GT.1.E-6) GO TO 3030                323
      GO TO 309                                                         324
3030  RI(K)=TAB(M+1,2,K)+(TAB(M,2,K)-TAB(M+1,2,K))*(Z(II)-TAB(M+1,1,K))/ 325
     1(TAB(M,1,K)-TAB(M+1,1,K))                                         326
      CALL DVCHK(NO)                                                    327
      IF (NO.EQ.2) GO TO 3031                                           328
      NO=3030                                                           329
      GO TO 940                                                         330
3031  CONTINUE                                                          331
      IF (M.EQ.JJJ) GO TO 3022                                          332
      IF (RI(K).GT.TAB(M+1,2,K).OR.RI(K).LT.TAB(M,2,K)) GO TO 309       333
3022  IF (MCOM.EQ.2) GO TO 305                                          334
      IF (KATCH.EQ.1) GO TO 304                                         335
      KATCH=1                                                           336
      GO TO 3050                                                        337
304   IF ((RX-RI(K)).GT.(RX-RM)) GO TO 309                              338
3050  RM=RI(K)                                                          339
      IF (K.EQ.2) GO TO 3040                                            340
      MPS=M                                                             341
      KATCHP=1                                                          342
      GO TO 309                                                         343
3040  MTS=M                                                             344
      KATCHT=1                                                          345
      GO TO 309                                                         346
305   CONTINUE                                                          347
```

```
      IF (KATCH.EQ.1) GO TO 306                                          348
      KATCH=1                                                            349
      GO TO 3050                                                         350
306   IF ((RI(K)-RX).GT.(RM-RX)) GO TO 309                               351
      GO TO 3050                                                         352
309   CONTINUE                                                           353
      RI(K)=RM                                                           354
      IF (KATCH.NE.0) GO TO 310                                          355
      RI(K)=RMAX+1.                                                      356
310   CONTINUE                                                           357
      K=3                                                                358
      IF (IRARF.EQ.1)RI(3)=RMAX+1.                                       359
      IF (IRARF.EQ.1) GO TO 315                                          360
      JJJ=NR-1                                                           361
      KATCH=0                                                            362
      DO 312 M=1,JJJ                                                     363
      RI(3)=RARF(M+1,2)+(RARF(M,2)-RARF(M+1,2))*(Z(II)-RARF(M+1,2))/(RAR 364
     1F(M,1)-RARF(M+1,1))                                                365
      NO=3122                                                            366
      CALL DVCHK(KQ)                                                     367
      IF (KQ.EQ.1) GO TO 940                                             368
      IF (M.EQ.JJJ.OR.M.EQ.1) GO TO 3122                                 369
      IF (RI(K).GT.RARF(M+1,2).OR.RI(K).LT.RARF(M,2)) GO TO 312          370
3122  IF (MCOM.EQ.2) GO TO 316                                           371
      IF (KATCH.EQ.1) GO TO 317                                          372
      KATCH=1                                                            373
      GO TO 3051                                                         374
317   IF ((RX-RI(K)).GT.(RX-RM)) GO TO 312                               375
3051  RM=RI(K)                                                           376
      MR=M                                                               377
      KATCHR=1                                                           378
      GO TO 312                                                          379
316   CONTINUE                                                           380
      IF (KATCH.EQ.1) GO TO 318                                          381
      KATCH=1                                                            382
      GO TO 3051                                                         383
```

```
318   IF ((RI(K)-RX).GT.(RM-RX)) GO TO 312                                  384
      GO TO 3051                                                            385
312   CONTINUE                                                              386
      RI(K)=RM                                                              387
      IF (KATCH.NE.0) GO TO 315                                             388
      RI(K)=RMAX+1.                                                         389
315   CONTINUE                                                              390
      KATCH=0                                                               391
      JJJ=NP-1                                                              392
      DO 3150 M=1,JJJ                                                       393
      IF ((SURF(M,1)-SURF(M+1,1)).GT.1.E-6) GO TO 3130                      394
      GO TO 3150                                                            395
3130  RI(4)=SURF(M+1,2)+(SURF(M,2)-SURF(M+1,2))*(Z(II)-SURF(M+1,1))/(SUR    396
     1F(M,1)-SURF(M+1,1))                                                   397
      NO=3130                                                               398
      CALL DVCHK(KQ)                                                        399
      IF (KQ.EQ.1) GO TO 940                                                400
      IF (M.EQ.JJJ) GO TO 3123                                              401
      IF (RI(4).GT.SURF(M+1,2).OR.RI(4).LT.SURF(M,2)) GO TO 3150            402
3123  IF (MCOM.EQ.2) GO TO 3105                                             403
      IF (KATCH.EQ.1) GO TO 3104                                            404
      KATCH=1                                                               405
      GO TO 3109                                                            406
3104  IF ((RX-RI(4)).GT.(RX-RM)) GO TO 3150                                 407
3109  RM=RI(4)                                                              408
      MS=M                                                                  409
      KATCHS=1                                                              410
      GO TO 3150                                                            411
3105  IF (KATCH.EQ.1) GO TO 3106                                            412
      KATCH=1                                                               413
      GO TO 3109                                                            414
3106  IF ((RI(4)-RX).GT.(RM-RX)) GO TO 3150                                 415
      GO TO 3109                                                            416
3150  CONTINUE                                                              417
      RI(4)=RM                                                              418
      IF (KATCH.NE.0) GO TO 3107                                            419
```

```
      RI(4)=RMAX+1.
3107  CONTINUE
      IF (KATCHP+KATCHT+KATCHR+KATCHS.EQ.0) GO TO 485
      ZI(1)=Z(II)
      ZI(2)=Z(II)
      ZI(3)=Z(II)
      ZI(4)=Z(II)
C         J LOOP FOR LEFT AND RIGHT R GRID LINES
C
      DO 700 J=1,2
      IF (J.EQ.2) GO TO 350
C
C         LEFT R GRID LINE
C
      JJ=J1
      KEY=0
      DO 340 N=1,4
      IF (R(J1).GT.RI(N)) GO TO 340
      IF (RI(N).GT.RX) GO TO 340
      IF (KEY.EQ.1) GO TO 330
      KEY=1
      KEEP=N
      GO TO 340
C
C         FIND CLOSEST
C
330   DIF1=RX-RI(KEEP)
      DIF2=RX-RI(N)
      IF (DIF1.LE.DIF2) GO TO 340
      KEEP=N
340   CONTINUE
      GO TO 375
C
C         RIGHT R GRID LINE
C
350   JJ=J2
```

```
      KEY=0                                                              456
      DO 360 N=1,4                                                       457
      IF (R(J2).LT.RI(N)) GO TO 360                                      458
      IF (RI(N).LT.RX) GO TO 360                                         459
      IF (KEY.EQ.1) GO TO 355                                            460
      KEY=1                                                              461
      KEEP=N                                                             462
      GO TO 360                                                          463
355   DIF1=RI(KEEP)-RX                                                   464
      DIF2=RI(N)-RX                                                      465
      IF (DIF1.LE.DIF2) GO TO 360                                        466
      KEEP=N                                                             467
360   CONTINUE                                                           468
375   IF (KEY.EQ.1) GO TO 400                                            469
C                                                                        470
C         NO POINTS BETWEEN RX AND GRID POINTS                           471
C                                                                        472
      ANS2(J,1)=Z(II)                                                    473
      ANS2(J,2)=R(JJ)                                                    474
      ANS2(J,3)=XMESH(II,JJ,1)                                           475
      ANS2(J,4)=XMESH(II,JJ,2)                                           476
      ANS2(J,5)=XMESH(II,JJ,3)                                           477
      ANS2(J,6)=XMESH(II,JJ,4)                                           478
      ANS2(J,7)=XMESH(II,JJ,5)                                           479
      ANS2(J,8)=XMESH(II,JJ,6)                                           480
      GO TO 700                                                          481
C                                                                        482
C         POINT FOUND BETWEEN RX AND GRID POINTS                         483
C                                                                        484
400   GO TO (405,410,470,481),KEEP                                       485
C                                                                        486
C         INTERSECTION ON PROJECTILE SHOCK                               487
C                                                                        488
405   N1=MPS                                                             489
      RY=RI(KEEP)                                                        490
      ZY=Z(II)                                                           491
```

```
      GO TO 520                                                          492
C                                                                        493
C          INTERSECTION ON TARGET SHOCK                                  494
C                                                                        495
410   N1=MTS                                                             496
      RY=RI(KEEP)                                                        497
      ZY=Z(II)                                                           498
      GO TO 520                                                          499
C                                                                        500
C        INTERSECTION ON RAREFACTION                                     501
C                                                                        502
470   N1=MR                                                              503
      RY=RI(KEEP)                                                        504
      ZY=Z(II)                                                           505
      GO TO 520                                                          506
C                                                                        507
C     INTERSECTION ON FREE SURFACE                                       508
C                                                                        509
481   N1=MS                                                              510
      RY=RI(KEEP)                                                        511
      ZY=Z(II)                                                           512
      GO TO 520                                                          513
C                                                                        514
C                                                                        515
485   WRITE (3,488) KEEP,II,JJ,ZX,RX                                     516
488   FORMAT (25HOERROR NEAR STATEMENT 485/1X5HKEEP=,I4,4X3HII=,I4,4X3HJ  517
     1J=,I4/1X3HZX=,E15.8,4X3HRX=,E15.8/1H1)                             518
      CALL EXIT                                                          519
C                                                                        520
C          FIND TABLE VALUES                                             521
C                                                                        522
520   IF (KEEP.EQ.3) GO TO 580                                           523
      IF (KEEP.EQ.4) GO TO 591                                           524
      DO 550 N=3,8                                                       525
      ANS2(J,N)=TAB(N1,N,KEEP)+(TAB(N1+1,N,KEEP)-TAB(N1,N,KEEP))*SQRT(((  526
     1RY-TAB(N1,2,KEEP))**2+(ZY-TAB(N1,1,KEEP))**2)/((TAB(N1+1,2,KEEP)-T  527
```

```
     1AB(N1,2,KEEP))**2+(TAB(N1+1,1,KEEP)-TAB(N1,1,KEEP))**2))                528
550   CONTINUE                                                                 529
      IF (ZX.LT.0..AND.ABS(RX-RADIUS).LT.1.E-6) GO TO 552                      530
      IF (RX.LT.RADIUS.OR.ABS(Z(II)).GT.1.E-6) GO TO 551                       531
552   CONTINUE                                                                 532
      ANS2(J,3)=0.                                                             533
      ANS2(J,6)=RHOSTR                                                         534
      ANS2(J,7)=0.                                                             535
      ANS2(J,8)=SQRT(BIGAPR/RHOSTR)                                            536
551   CONTINUE                                                                 537
      GO TO 600                                                                538
580   DO 590 N=3,8                                                             539
      ANS2(J,N)=RARF(N1,N)+(RARF(N1+1,N)-RARF(N1,N))*SQRT(((RY-RARF(N1,2       540
     1))**2+(ZY-RARF(N1,1))**2)/((RARF(N1+1,2)-RARF(N1,2))**2+(RARF(N1+1       541
     1,1)-RARF(N1,1))**2))                                                     542
590   CONTINUE                                                                 543
      GO TO 600                                                                544
591   DO 592 N=3,8                                                             545
      ANS2(J,N)=SURF(N1,N)+(SURF(N1+1,N)-SURF(N1,N))*SQRT(((RY-SURF(N1,2       546
     1))**2+(ZY-SURF(N1,1))**2)/((SURF(N1+1,2)-SURF(N1,2))**2+(SURF(N1+1       547
     1,1)-SURF(N1,1))**2))                                                     548
592   CONTINUE                                                                 549
600   CALL DVCHK(NO)                                                           550
      IF (NO.EQ.2) GO TO 605                                                   551
      NO=600                                                                   552
      GO TO 940                                                                553
605   ANS2(J,1)=ZY                                                             554
      ANS2(J,2)=RY                                                             555
C                                                                              556
C         END OF LOOP FOR BOTH R GRID LINES                                    557
C                                                                              558
700   CONTINUE                                                                 559
C                                                                              560
C         INTERPOLATE FOR UPPER AND LOWER VALUES                               561
C                                                                              562
      DO 720 J=3,8                                                             563
```

```
      ANS1(I,J)=ANS2(1,J)+(ANS2(2,J)-ANS2(1,J))*(RX-ANS2(1,2))/(ANS2(2,2
     1)-ANS2(1,2))
720   CONTINUE
      CALL DVCHK(NO)
      IF (NO.EQ.2) GO TO 730
      NO=720
      GO TO 940
730   ANS1(I,1)=Z(II)
      ANS1(I,2)=RX
C
C         END OF LOOP FOR BOTH Z GRID LINES
C
800   CONTINUE
C
C         FIND FINAL VALUES
C
      DO 810 J=1,6
      ANS(J)=ANS1(1,J+2)+(ANS1(2,J+2)-ANS1(1,J+2))*(ZX-ANS1(1,1))/(ANS1(
     12,1)-ANS1(1,1))
810   CONTINUE
      CALL DVCHK(NO)
      IF (NO.EQ.2) GO TO 820
      NO=810
      GO TO 940
820   RETURN
C
940   WRITE (3,942)
942   FORMAT (35HODIVIDE CHECK ERROR IN SUBR. DBLTRP)
950   WRITE (3,952) NO,ZX,RX,I1,J1,KEEP,ZI,RI
952   FORMAT (19H NEAR STATEMENT NO.,I4/1X3HZX=,E15.8,4X3HRX=,E15.8/1X3H
     1I1=,I4,4X3HJ1=,I4,4X5HKEEP=,I4/1X3HZI=,4E18.8/1X3HRI=,4E18.8)
      WRITE (3,955) ((ANS1(I,J),J=1,8),I=1,2),((ANS2(I,J),J=1,8),I=1,2)
955   FORMAT (1X5HANS1=/8E16.8/8E16.8/1X5HANS2=/8E16.8/8E16.8/1H1)
      XYZ=-2.
      ZYX=SQRT(XYZ)
      CALL EXIT
```

```
      RETURN                                                                 600
      DEBUG SUBCHK
      END                                                                    601
      SUBROUTINE SHOCK                                                         1
C         COMPUTES SHOCK VALUES                                              2
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPS     3
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LEN     4
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS            5
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2      6
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(     7
     115,4),RPART(15,2)                                                      8
C                                                                            9
      COMMON ZO,RO,PO,UO,VO,LO,MO,RHOO,EO,AO,UBARO,VBARO                     10
C                                                                           11
      COMMON NP,NT,NR,NI,NDEL,ISUB                                          12
C                                                                           13
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                       14
      COMMON DIRCOS                                                         15
      COMMON TIME                                                           16
      COMMON IRARF                                                          17
      COMMON KSTOP                                                          18
      COMMON TPSI                                                           19
      COMMON KKK                                                            20
C                                                                           21
      REAL LO,MO,LENGTH,MU,KO                                               22
C                                                                           23
      DIMENSION ANS(6)                                                      24
      EXTERNAL FGOF1                                                        25
      EPS=.0000001                                                          26
C                                                                           27
C         BEGIN SHOCK POINT COMPUTATIONS                                    28
      DO 505 K=1,2                                                          29
      GO TO (158,160),K                                                     30
158   NN=NP                                                                 31
      GO TO 162                                                             32
160   NN=NT                                                                 33
```

```
      VBARS=0.                                                          34
162   DO 500 I=1,NN                                                     35
      MPROJ=0                                                           36
      IF (TAB(I,14,K).LT.0.) GO TO 500                                  37
164   CONTINUE                                                          38
C                                                                       39
C         INITIALIZE TO ITERATE ON 1 SHOCK POINT                        40
C                                                                       41
      NBIC=0                                                            42
      ZO=TAB2(I,1,K)                                                    43
      RO=TAB2(I,2,K)                                                    44
      PO=TAB(I,3,K)                                                     45
      UO=TAB(I,4,K)                                                     46
      VO=TAB(I,5,K)                                                     47
      RHOO=TAB(I,6,K)                                                   48
      EO=TAB(I,7,K)                                                     49
      AO=TAB(I,8,K)                                                     50
      LO=TAB(I,9,K)                                                     51
      MO=TAB(I,10,K)                                                    52
      UBARO=TAB(I,11,K)                                                 53
      VBARO=TAB(I,12,K)                                                 54
      UTOH=TAB(I,13,K)                                                  55
      UTO=UTOH                                                          56
      ITS44=ITS4                                                        57
      IF (IRARF.EQ.1) GO TO 170                                         58
      M=1-(NR-2)*(K-2)                                                  59
      FF=RO-RARF2(M+1,2)-(RARF2(M,2)-RARF2(M+1,2))*(ZO-RARF2(M+1,1))/(RA 60
     1RF2(M,1)-RARF2(M+1,1))                                            61
169   CONTINUE                                                          62
      IF (FF.LT..001) GO TO 350                                         63
170   IF (I.NE.1) GO TO 180                                             64
      MO=TAB(I,10,K)                                                    65
      LO=0.                                                             66
      GO TO 190                                                         67
180   IF (I.LT.NN) GO TO 184                                            68
      UP=H*(TAB(I-1,13,K)-TAB(I,13,K))                                  69
```

```
      TMP=SQRT((TAB2(I-1,2,K)-RO)**2+(TAB2(I-1,1,K)-ZO)**2)                70
      DOMEG=UP/TMP                                                         71
      GO TO 186                                                            72
184   DR1=TAB2(I+1,2,K)-RO                                                 73
      DR2=RO-TAB2(I-1,2,K)                                                 74
      DZ1=TAB2(I+1,1,K)-ZO                                                 75
      DZ2=ZO-TAB2(I-1,1,K)                                                 76
      UP=H*(TAB(I-1,13,K)-TAB(I+1,13,K))                                   77
      TMP=SQRT(DR1**2+DZ1**2)+SQRT(DR2**2+DZ2**2)                          78
      DOMEG=UP/TMP                                                         79
      KICK=184                                                             80
      CALL DVCHK(KQ)                                                       81
      IF (KQ.EQ.1) GO TO 9980                                              82
C                                                                          83
C         COMPUTE NEW LO,MO                                                84
C                                                                          85
186   COMEG=COS(DOMEG)                                                     86
      SOMEG=SIN(DOMEG)                                                     87
      XLO=LO*COMEG+MO*SOMEG                                                88
      XMO=MO*COMEG-LO*SOMEG                                                89
      LO=XLO                                                               90
      MO=XMO                                                               91
      IF (PO.GT..0025) GO TO 190                                           92
      PO=0.                                                                93
      UO=0.                                                                94
      VO=VP*(1.-(-1.)**K)/2.                                               95
      RHOO=RHOSTR                                                          96
      EO=0.                                                                97
      AO=SQRT(BIGAPR/RHOSTR)                                               98
      UBARO=MO*VO                                                          99
      VBARO=VBARS                                                         100
      UTO=UBARO+AO*(-1.)**K                                               101
      GO TO 350                                                           102
190   ITS33=ITS3                                                          103
C                                                                         104
C         FIND GUESS TO START ITERATION                                   105
```

```
C                                                                       106
195   CALL GUESS(1,KOD2,ZO,RO,I,K,ZZ,RR,DZ,DR)                         107
      IF (KOD2.EQ.1) GO TO 200                                          108
      WRITE (3,198) I,K,ZO,RO                                           109
      WRITE (3,7002) ZZ,RR,DZ,DR                                        110
198   FORMAT (31HONO GUESS FOUND FOR SHOCK POINT/3HOI=,I4,6X2HK=,I4,10X3 111
     1HZO=,E15.8,10X3HRO=,E15.8/1H1)                                    112
      CALL EXIT                                                         113
200   CONTINUE                                                          114
      KY=K                                                              115
      NTW=0                                                             116
      IF (K.EQ.2) GO TO 201                                             117
      VBARS=VP*LO                                                       118
      DIRCOS=-MO                                                        119
      GO TO 203                                                         120
201   DIRCOS=MO                                                         121
203   CONTINUE                                                          122
      CALL NRIT2(Z1,R1,ZZ,DZ,RR,DR,EPS1,EPS2,FGOF1,ITS1,KODE)           123
      IF (KODE.EQ.0) GO TO 205                                          124
C                                                                       125
C     BICHARACTERISTIC SELECTION SCHEME                                 126
C                                                                       127
      IF (NBIC.EQ.0) GO TO 204                                          128
      IF (NTW.EQ.8)KY=KY+1                                              129
      IF (NTW.GT.21) GO TO 7000                                         130
      ANG1=ANG1+DTPSI*(-1.)**KY                                         131
      DIRCOS=SIN(ANG1)                                                  132
      LO=COS(ANG1)                                                      133
      NTW=NTW+1                                                         134
      GO TO 203                                                         135
204   CONTINUE                                                          136
      NBIC=1                                                            137
      CALL DBLTRP(ZZ,RR,ANS)                                            138
      UA=ANS(2)                                                         139
      VA=ANS(3)                                                         140
      AA=ANS(6)                                                         141
```

```
      ZZZ=ZZ+DZ                                                          142
      CALL DBLTRP(ZZZ,RR,ANS)                                            143
      UB=ANS(2)                                                          144
      VB=ANS(3)                                                          145
      AB=ANS(6)                                                          146
      RRR=RR+DR                                                          147
      CALL DBLTRP(ZZ,RRR,ANS)                                            148
      UC=ANS(2)                                                          149
      VC=ANS(3)                                                          150
      AC=ANS(6)                                                          151
      MM=0                                                               152
      TPSI=1.5708*(-1.)**K                                               153
      XB=ZZ                                                              154
      YB=RR                                                              155
      NOM=5                                                              156
      CA=NOM                                                             157
6201  DTPSI=.01745                                                       158
      DTPSI=CA*DTPSI                                                     159
      TPSI=TPSI+DTPSI*(-1.)**K                                           160
      A1=1.+H*(VB-VA+(AB-AA)*SIN(TPSI))/DZ                               161
      B1=H*(VC-VA+(AC-AA)*SIN(TPSI))/DR                                  162
      C1=-(ZZ-ZO+H*(VA+AA*SIN(TPSI)))                                    163
      A2=H*(UB-UA+(AB-AA)*COS(TPSI))/DZ                                  164
      B2=1.+H*(UC-UA+(AC-AA)*COS(TPSI))/DR                               165
      C2=-(RR-RO+H*(UA+AA*COS(TPSI)))                                    166
      DET=A1*B2-A2*B1                                                    167
      DELX=(B2*C1-B1*C2)/DET                                             168
      DELY=(A1*C2-A2*C1)/DET                                             169
C                                                                        170
C          TEST FOR SAME REGION                                          171
C                                                                        172
      XB1=XB+DELX                                                        173
      YB1=YB+DELY                                                        174
      M=COMP(XB,YB,XB1,YB1)                                              175
      IF (M.EQ.1) GO TO 6203                                             176
      MM=MM+1                                                            177
```

```
      IF (MM.LT.360/NOM) GO TO 6201                                          178
7000  WRITE (3,7001)                                                         179
7001  FORMAT (41HOBICHARACTERISTIC SELECTION SCHEME FAILED)                  180
      WRITE (3,614) ZO,RO                                                    181
614   FORMAT (1X5HZO  =,E15.8,4X5HRO  =,E15.8)                               182
      WRITE (3,7002) UA,VA,AA,UB,VB,AB,UC,VC,AC,ZZ,DZ,RR,DR,ANG1             183
7002  FORMAT (4E16.8)                                                        184
      CALL EXIT                                                              185
6203  CONTINUE                                                               186
      WRITE (3,6210)                                                         187
6210  FORMAT (53HOBICHARACTERISTIC SELECTION SCHEME EMPLOYED BY SHOCKX)       188
6204  ANG1=TPSI                                                              189
      T1=DIRCOS                                                              190
      T2=LO                                                                  191
      DIRCOS=SIN(ANG1)                                                       192
      LO=COS(ANG1)                                                           193
      GO TO 203                                                              194
205   CONTINUE                                                               195
      UBARS1=0.                                                              196
      IF (K.EQ.1)UBARS1=MO*VP                                                197
      CALL DBLTRP(Z1,R1,ANS)                                                 198
      P1=ANS(1)                                                              199
      U1=ANS(2)                                                              200
      V1=ANS(3)                                                              201
      RHO1=ANS(4)                                                            202
      E1=ANS(5)                                                              203
      A1=ANS(6)                                                              204
206   CONTINUE                                                               205
      KICK=205                                                               206
      CALL DVCHK(KQ)                                                         207
      IF (KQ.EQ.1) GO TO 9980                                                208
      IF (NBIC.EQ.0) GO TO 207                                               209
7003  SINTH=ABS(DIRCOS)                                                      210
      COSTH=ABS(LO)                                                          211
      DIRCOS=T1                                                              212
      LO=T2                                                                  213
```

```
207   CONTINUE                                                          214
      UBAR1=LO*U1+MO*V1                                                 215
      IF (K.EQ.2) GO TO 208                                             216
      IF (UBAR1.LT.VP/2.) GO TO 208                                     217
      UBAR1=UBARS1-UBAR1                                                218
208   CONTINUE                                                          219
218   CONTINUE                                                          220
      M1=PART(1,Z1,R1,ZZ,RR,DELTA,NDEL)                                 221
      IF (M1.EQ.1) GO TO 210                                            222
      PUR=0.                                                            223
      PVR=0.                                                            224
      GO TO 215                                                         225
210   CALL DBLTRP(ZZ,RR,ANS)                                            226
      DP=RR-R1                                                          227
      PUR=(ANS(2)-U1)/DP                                                228
219   CONTINUE                                                          229
      PVR=(ANS(3)-V1)/DP                                                230
215   M1=PART(2,Z1,R1,ZZ,RR,DELTA,NDEL)                                 231
      IF (M1.EQ.1) GO TO 220                                            232
      PUZ=0.                                                            233
      PVZ=0.                                                            234
      PAZ=0.                                                            235
      GO TO 225                                                         236
220   CALL DBLTRP(ZZ,RR,ANS)                                            237
      DP=ZZ-Z1                                                          238
      PUZ=(ANS(2)-U1)/DP                                                239
      PVZ=(ANS(3)-V1)/DP                                                240
225   CONTINUE                                                          241
      IF (NBIC.EQ.1) GO TO 7004                                         242
      PURB1=LO*PUR+MO*PVR                                               243
      PVRB1=LO*PVR-MO*PUR                                               244
      PVZB1=LO*PVZ-MO*PUZ                                               245
      PVEB1=-MO*PVRB1+LO*PVZB1                                          246
      SBAR1=PVEB1                                                       247
226   CONTINUE                                                          248
      IF (ABS(R1).LE.EPS) GO TO 235                                     249
```

```
      SBAR1=SBAR1+U1/R1                                                      250
      GO TO 240                                                              251
7004  CONTINUE                                                               252
      IF (V1.GT.VP/2..AND.K.EQ.1)V1=VP-V1                                    253
      SBAR1=SINTH**2*PUR-SINTH*COSTH*(PUZ+PVR)+COSTH**2*PVZ                  254
      GO TO 226                                                              255
235   SBAR1=SBAR1+PUR                                                        256
C                                                                            257
C                                                                            258
240   ITS22=ITS2                                                             259
      MMM=0                                                                  260
250   CONTINUE                                                               261
      CALL EQOSS(PFR,PFE)                                                    262
      BIGA1=RHO1*A1                                                          263
      KICK=250                                                               264
      CALL DVCHK(KQ)                                                         265
      IF (KQ.EQ.1) GO TO 9980                                                266
      TEMP=1.-RHOSTR/RHO0                                                    267
      TMP=SQRT(P0*TEMP/RHOSTR)                                               268
      IF (K.EQ.2) GO TO 251                                                  269
      IF (TMP.LT.VP/2.) GO TO 251                                            270
      MPROJ=1                                                                271
      TMP=UBARS1-TMP                                                         272
251   CONTINUE                                                               273
      TMP6=TMP                                                               274
256   CONTINUE                                                               275
      FNBIC=NBIC                                                             276
      TMP1=P1+BIGA1*UBAR1-RHO1*H*SBAR1*A1**2+BIGA1*(-UBAR1+COSTH*U1+SINT     277
     1H*V1-SINTH*L0*VBARS+COSTH*M0*VBARS)*(FNBIC)                            278
      TMP2=PFR*TEMP+P0*RHOSTR/RHO0**2                                        279
      TMP5=PFE*TEMP                                                          280
      GTMP=-RHO1*A1                                                          281
      IF (NBIC.EQ.0) GO TO 259                                               282
      GTMP=GTMP*(COSTH*L0+SINTH*M0)                                          283
259   CONTINUE                                                               284
      BIGG=P0-(GTMP*TMP+TMP1)                                                285
```

```
      PGR=PFR-((GTMP*TMP2)/(2.*RHOSTR*TMP6))                            286
      PGE=PFE-((GTMP*TMP5)/(2.*RHOSTR*TMP6))                            287
265   TMP=.5*(1./RHOSTR-1./RHO0)                                        288
      BIGH=E0-TMP*P0                                                    289
      PHR=-TMP*PFR-.5*P0/RHO0**2                                        290
      PHE=1.-TMP*PFE                                                    291
      IF (ABS(BIGH).GT..0001) GO TO 267                                 292
      BIGH=0.                                                           293
267   IF (ABS(BIGG).GT..0001) GO TO 269                                 294
      BIGG=0.                                                           295
269   CONTINUE                                                          296
C                                                                       297
C          COMPUTE DELTA E0,DELTA RHO0                                  298
C                                                                       299
      DOWN=PGE*PHR-PGR*PHE                                              300
      DE0=(-BIGG*PHR+BIGH*PGR)/DOWN                                     301
      DRHO0=(-BIGH*PGE+BIGG*PHE)/DOWN                                   302
C                                                                       303
      E02=E0+DE0                                                        304
      IF (E02.LT.0.)E02=0.                                              305
      RHO02=RHO0+DRHO0                                                  306
      IF (RHO02.LT.RHOSTR)RHO02=RHO0                                    307
      KICK=265                                                          308
      CALL DVCHK(KQ)                                                    309
      IF (KQ.EQ.1) GO TO 9980                                           310
      CALL EQOSP(RHO02,E02,P02)                                         311
      UBAR02=(1.-RHOSTR/RHO02)*(P02/RHOSTR)                             312
      IF (UBAR02.GT.0.) GO TO 2669                                      313
      WRITE (3,2700) P02,RHO02,E02,R0,Z0                                314
      WRITE (3,7002) P1,U1,V1,RHO1,E1,Z1TR1,SBAR1                       315
2700  FORMAT (4E16.8)                                                   316
2669  CONTINUE                                                          317
      UBAR02=SQRT(UBAR02)                                               318
      IF (E02.LT.1.E-5) GO TO 273                                       319
      IF (ABS((E02-E0)/E02).LT.EPS4) GO TO 273                          320
      IF (ABS(DE0).GT..01*EPS4) GO TO 275                               321
```

```
273   IF (ABS((RHO02-RHO0)/RHO02).LE.EPS3) GO TO 285                    322
      IF (ABS(DRHO0).LT.EPS3) GO TO 285                                 323
275   ITS22=ITS22-1                                                     324
      IF (ITS22.GT.0) GO TO 280                                         325
      WRITE (3,278) ITS2                                                326
278   FORMAT (35HOE AND RHO FAILED TO CONVERGE AFTER,I4,6H TRIES)       327
      WRITE (3,279) I,K,Z0,R0,E0,RHO0,P0,E02,RHO02,P02                  328
279   FORMAT (1X2HI=,I4,4X2HK=,I4/1X4HZ0 =,E15.8,4X4HR0 =,E15.8,4X4HE0 = 329
     1,E15.8,4X6HRHO0 =,E15.8,4X4HP0 =,E15.8/1X4HE02=,E15.8,4X6HRHO02=,E 330
     115.8,4X4HP02=,E15.8/1H1)                                          331
      STOP                                                              332
280   E0=E02                                                            333
      RHO0=RHO02                                                        334
      P0=P02                                                            335
      UBAR0=UBAR02                                                      336
      GO TO 250                                                         337
285   E0=E02                                                            338
      RHO0=RHO02                                                        339
      UBAR0=UBAR02                                                      340
      A0=SQRT(PFR+P02*PFE/RHO0**2)                                      341
C                                                                       342
C                                                                       343
      CALL DVCHK(KQ)                                                    344
      KICK=285                                                          345
      IF (KQ.EQ.1) GO TO 9980                                           346
      IF (K.EQ.2) GO TO 286                                             347
      VBAR0=VP*L0                                                       348
      GO TO 287                                                         349
286   VBAR0=0.                                                          350
287   CONTINUE                                                          351
295   P0=P02                                                            352
      IF (K.EQ.2) GO TO 296                                             353
      UBAR0=UBARS1-UBAR0                                                354
296   CONTINUE                                                          355
      UT0=(RHO0*UBAR0-RHOSTR*UBARS1)/(RHO0-RHOSTR)                      356
      V0=M0*UBAR0+L0*VBAR0                                              357
```

```
      UO=LO*UBARO-MO*VBARO                                                    358
      UBAR=.5*(UTOH+UTO)                                                      359
      IF (ABS((UBAR-UTOH)/UBAR).LE.EPS6) GO TO 350                            360
      IF (ABS(UBAR-UTOH).LT.EPS6) GO TO 350                                   361
      ITS44=ITS44-1                                                           362
      IF (ITS44.GT.0) GO TO 325                                               363
      WRITE (3,297) ITS4,UTOH,UTO                                             364
297   FORMAT (30HOUBAR FAILED TO CONVERGE AFTER,I4,6H TRIES/1X5HUTOH=,E1      365
     15.8,4X4HUTO=,E15.8)                                                     366
      CALL EXIT                                                               367
      WRITE (3,279) I,K,ZO,RO,EO,RHOO,PO,EO2,RHOO2,PO2                        368
C                                                                             369
C         INIT. FOR MORE U BAR ITERATIONS                                     370
C                                                                             371
325   UTOH=UBAR                                                               372
      AVMO=(TAB(I,10,K)+MO)*.5                                                373
      AVLO=(TAB(I,9,K)+LO)*.5                                                 374
      ZO=TAB(I,1,K)+UBAR*H*AVMO-VBARS*AVLO*H                                  375
      RO=TAB(I,2,K)+UBAR*H*AVLO-VBARS*AVMO*H                                  376
      LO=AVLO                                                                 377
      MO=AVMO                                                                 378
      GO TO 195                                                               379
C                                                                             380
C         ONE SHOCK POINT HAS CONVERGED                                       381
C                                                                             382
350   TAB2(I,1,K)=ZO                                                          383
      TAB2(I,2,K)=RO                                                          384
      TAB2(I,3,K)=PO                                                          385
      TAB2(I,4,K)=UO                                                          386
      TAB2(I,5,K)=VO                                                          387
      TAB2(I,6,K)=RHOO                                                        388
      TAB2(I,7,K)=EO                                                          389
      TAB2(I,8,K)=AO                                                          390
      TAB2(I,9,K)=LO                                                          391
      TAB2(I,10,K)=MO                                                         392
      TAB2(I,11,K)=UBARO                                                      393
```

```
      TAB2(I,12,K)=VBARO                                                    394
      TAB2(I,13,K)=UTO                                                      395
      KICK=500                                                              396
      CALL DVCHK(KQ)                                                        397
      IF (KQ.EQ.1) GO TO 9980                                               398
C                                                                           399
C                                                                           400
500   CONTINUE                                                              401
505   CONTINUE                                                              402
      RETURN                                                                403
9980  WRITE (3,9985) KICK                                                   404
9985  FORMAT (32HODIVIDE CHECK NEAR STATEMENT NO.,I5,15H IN SUBR. SHOCK/    405
     11H1)                                                                  406
      CALL EXIT                                                             407
      RETURN                                                                408
      END                                                                   409
      SUBROUTINE FGOF1(ZX,RX,SS,QQ)                                           1
C                                                                             2
C         COMPUTES S1,Q1 FOR SHOCK LINE                                       3
C         ITERATION FOR Z1,R1                                                 4
C                                                                             5
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPS      6
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LEN      7
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS             8
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2       9
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(     10
     115,4),RPART(15,2)                                                      11
C                                                                            12
C                                                                            13
      COMMON ZO,RO,PO,UO,VO,LO,MO,RHOO,EO,AO,UBARO,VBARO                     14
C                                                                            15
      COMMON NP,NT,NR,NI,NDEL,ISUB                                           16
C                                                                            17
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                        18
      COMMON DIRCOS                                                          19
      COMMON TIME                                                            20
```

```
      COMMON IRARF                                                      21
      COMMON KSTOP                                                      22
      COMMON TPSI                                                       23
      COMMON KKK                                                        24
      REAL LO,MO,LENGTH,MU,KO                                           25
C                                                                       26
      DIMENSION ANS(6)                                                  27
C                                                                       28
      REAL LO,MO                                                        29
      CALL DBLTRP(ZX,RX,ANS)                                            30
      U1=ANS(2)                                                         31
      V1=ANS(3)                                                         32
      A1=ANS(6)                                                         33
      SS=ZX-ZO+H*(V1+A1*DIRCOS)                                         34
      QQ=RX-RO+H*(U1+A1*LO)                                             35
      RETURN                                                            36
      END                                                               37
      SUBROUTINE INTER(PHI,NMAX,MMAX)
C     SUBROUTINE INTER
C         COMPUTES INTERIOR REGION POINTS                              2
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPS  3
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LEN  4
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS         5
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2   6
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(  7
     115,4),RPART(15,2)                                                   8
C                                                                         9
C                                                                        10
      COMMON ZO,RO,PO,UO,VO,LO,MO,RHOO,EO,AO,UBARO,VBARO                 11
C                                                                        12
      COMMON NP,NT,NR,NI,NDEL,ISUB                                       13
C                                                                        14
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                    15
      COMMON DIRCOS                                                      16
      COMMON TIME                                                        17
      COMMON IRARF                                                       18
```

```
      COMMON KSTOP                                                          19
      COMMON TPSI                                                           20
      COMMON KKK                                                            21
C                                                                           22
C                                                                           23
      REAL LO,MO,LENGTH,MU,KO                                               24
      DOUBLE PRECISION PHI(20,20,6)
      DIMENSION ANS(6),LL(3),ZI(11),RI(11),PI(11),UI(11),VI(11),RHOI(11)    25
     1,EI(11),AI(11),PUR(11),PVR(11),PAR(11),PUZ(11),PVZ(11),PAZ(11),PSI    26
     1(7),SPSI(11),CPSI(11),S(11)                                           27
C                                                                           28
      EXTERNAL FGOFI,FGOF5                                                  29
      INTEGER CHECK,CHECK2                                                  30
C                                                                           31
1     FORMAT (1H1)                                                          32
      TSSS=1.1
      EPS=.0000001                                                          33
      DO 905 J=1,20                                                         34
      DO 900 I=1,20                                                         35
      M=TEST(Z(I),R(J))                                                     36
      ZO=Z(1)                                                               37
      RO=R(J)                                                               38
      KICK=1                                                                39
      CALL DVCHK(KQ)                                                        40
      IF (KQ.EQ.1) GO TO 9980                                               41
      IF (M.EQ.3.AND.ZO.LT.EPS.AND.ABS(RO-RADIUS).LT.EPS)M=1                42
      IF (M.NE.1) GO TO 900                                                 43
      DO 2 L=1,NP                                                           44
      IF (TAB(L,14,1).LT.0.) GO TO 20                                       45
2     CONTINUE                                                              46
      IF (IRARF.EQ.1) GO TO 20                                              47
      DO 3 N=1,NR                                                           48
      IF (R(J).GT.RARF(N,2)) GO TO 5                                        49
3     CONTINUE                                                              50
      GO TO 6                                                               51
5     CONTINUE                                                              52
```

```
      M=PICK(Z(I),R(J),3)                                                    53
      FF=R(J)-RARF(M+1,2)-(RARF(M,2)-RARF(M+1,2))*(Z(I)-RARF(M+1,1))/(RA      54
     1RF(M,1)-RARF(M+1,1))                                                   55
      IF (FF.GT.0.) GO TO 20                                                 56
6     CONTINUE                                                               57
      PO2=RARF(1,3)                                                          58
      UO2=RARF(1,4)                                                          59
      VO2=RARF(1,5)                                                          60
      RHOO2=RARF(1,6)                                                        61
      EO2=RARF(1,7)                                                          62
      AO2=RARF(1,8)                                                          63
      GO TO 870                                                              64
20    CONTINUE                                                               65
      CALL GUESS(2,KOD,ZO,RO,I,J,ZZ,RR,DZ,DR)                               66
      IF (KOD.EQ.1) GO TO 580                                                67
      WRITE (3,575) I,J,ZO,RO                                                68
575   FORMAT (41HONO GUESS FOUND FOR INTERIOR REGION POINT/3HOI=I4,6X2HK      69
     1=,I4,10X3HZO=,E15.8,10X3HRO=,E15.8/1H1)                                70
      CALL EXIT                                                              71
  580 IF(TIME.GT.TSSS) GO TO 150
      CALL DBLTRP(ZZ,RR,ANS)
      GO TO 151
  150 CALL  DSURFT(ZZ,RR,ANS,PHI,NMAX,MMAX)
C                                                                            73
C          INITIALIZE FOR 1 POINT                                            74
C                                                                            75
C     PSI(1)=0.
  151 PSI(1)=0.
      PSI(3)=1.0472                                                          77
      PSI(4)=2.0944                                                          78
      PSI(6)=4.18879                                                         79
      PSI(7)=5.23599                                                         80
      NBIC=0                                                                 81
      ITI22=ITI2                                                             82
      PO=ANS(1)                                                              83
      UO=ANS(2)                                                              84
```

```
      V0=ANS(3)                                                          85
      RHO0=ANS(4)                                                        86
      E0=ANS(5)                                                          87
      A0=ANS(6)                                                          88
      KICK=580                                                           89
      CALL DVCHK(KQ)                                                     90
      IF (KQ.EQ.1) GO TO 9980                                            91
590   IF (ABS(R0).GT.EPS) GO TO 594                                      92
      L1=1                                                               93
      LL(1)=4                                                            94
      LL(2)=6                                                            95
      LLL=2                                                              96
      GO TO 620                                                          97
C                                                                        98
C                                                                        99
594   IF (ABS(R0-RADIUS).GT.EPS) GO TO 600                              100
      IF (Z0.GT.EPS) GO TO 610                                          101
      L1=2                                                              102
      LL(1)=3                                                           103
      LL(2)=7                                                           104
      LLL=2                                                             105
      GO TO 620                                                         106
600   IF (R0.LE.RADIUS.OR.ABS(Z0).GT.EPS) GO TO 610                     107
      L1=3                                                              108
      LL(1)=6                                                           109
      LL(2)=7                                                           110
      LLL=2                                                             111
      GO TO 620                                                         112
C                                                                       113
C                                                                       114
610   L1=4                                                              115
      LL(1)=1                                                           116
      LL(2)=4                                                           117
      LL(3)=6                                                           118
      LLL=3                                                             119
C                                                                       120
```

```
C          ITERATE FOR I VALUES                                                 121
C                                                                               122
619    CONTINUE                                                                 123
620    DO 630 KK=1,LLL                                                          124
       LUMP=L1                                                                  125
       ISUB=LL(KK)                                                              126
621    CONTINUE                                                                 127
       TPSI=PSI(ISUB)                                                           128
       SPSI(ISUB)=SIN(TPSI)                                                     129
       CPSI(ISUB)=COS(TPSI)                                                     130
C      CALL NRIT2(ZI(ISUB),RI(ISUB),ZZ,DZ,RR,DR,EPI1,EPI2,FGOFI,ITI1,KODE      131
C     1)
       CALL NRIT2(ZI(ISUB),RI(ISUB),ZZ,DZ,RR,DR,EPI1,EPI2,FGOFI,ITI1,KODE      131
      1,PHI,NMAX,MMAX)
       IF (KODE.NE.0) GO TO 6200                                                133
  625  IF(TIME.GT.TSSS) GO TO 125
       CALL DBLTRP(ZI(ISUB),RI(ISUB),ANS)
       GO TO 122
  125  CALL DSURFT(ZI(ISUB),RI(ISUB),ANS,PHI,NMAX,MMAX)
  122  PI(ISUB)=ANS(1)
C      PI(ISUB)=ANS(1)
       UI(ISUB)=ANS(2)                                                          136
       VI(ISUB)=ANS(3)                                                          137
       RHOI(ISUB)=ANS(4)                                                        138
       EI(ISUB)=ANS(5)                                                          139
       AI(ISUB)=ANS(6)                                                          140
630    CONTINUE                                                                 141
C                                                                               142
C                                                                               143
       KICK=630                                                                 144
       CALL DVCHK(KQ)                                                           145
       IF (KQ.EQ.1) GO TO 9980                                                  146
       GO TO 6400                                                               147
C                                                                               148
C     BICHARACTERISTIC SELECTION SCHEME                                         149
C                                                                               150
```

```
7000  WRITE (3,7001)                                                    151
7001  FORMAT (41HOBICHARACTERISTIC SELECTION SCHEME FAILED)             152
      WRITE (3,614) ZO,RO                                               153
      WRITE (3,7002) (PSI(MNMN),MNMN=1,7),UA,VA,AA,UB,VB,AB,UC,VC,AC,ZZ,  154
     1DZ,RR,DR,ANG1,ANG2                                                155
      SUB=ISUB                                                          156
      WRITE (3,7002) SUB,ZI(ISUB),RI(ISUB)                              157
7002  FORMAT (4E16.8)                                                   158
      CALL EXIT                                                         159
6200  CONTINUE                                                          160
      IF (NBIC.NE.0) GO TO 7000                                         161
      IF (L1.NE.2.OR.LL(1).EQ.1) GO TO 7300                             162
      LL(1)=1                                                           163
      GO TO 619                                                         164
7300  CONTINUE                                                          165
      IF (L1.NE.3) GO TO 7310                                           166
      IF (PSI(6).GT.4.2) GO TO 7310                                     167
      PSI(6)=5.75959                                                    168
      GO TO 619                                                         169
7310  CONTINUE                                                          170
      IF(TIME.GT.TSSS) GO TO 222
      CALL DBLTRP(ZZ,RR,ANS)
      GO TO 223
  222 CALL DSURFT(ZZ,RR,ANS,PHI,NMAX,MMAX)
  223 UA=ANS(2)
C     UA=ANS(2)
      VA=ANS(3)                                                         173
      AA=ANS(6)                                                         174
      ZZZ=ZZ+DZ                                                         175
      IF(TIME.GT.TSSS) GO TO 128
      CALL DBLTRP(ZZZ,RR,ANS)
      GO TO 129
  128 CALL DSURFT(ZZZ,RR, ANS,PHI,NMAX,MMAX)
  129 UB=ANS(2)
C     UB=ANS(2)
      VB=ANS(3)                                                         178
```

```
      AB=ANS(6)                                                          179
      RRR=RR+DR                                                          180
      IF(TIME.GT.TSSS) GO TO 130
      CALL DBLTRP(ZZ,RRR,ANS)
      GO TO 131
  130 CALL DSURFT(ZZ,RRR,ANS,PHI,NMAX,MMAX)
  131 UC=ANS(2)
C     UC=ANS(2)
      VC=ANS(3)                                                          183
      AC=ANS(6)                                                          184
      MM=0                                                               185
      TPSI=PSI(ISUB)                                                     186
      XB=ZZ                                                              187
      YB=RR                                                              188
      NOM=5                                                              189
      CA=NOM                                                             190
      DO 6210 LM=1,2                                                     191
 6201 DTPSI=.01745                                                       192
      DTPSI=CA*DTPSI                                                     193
      TPSI=TPSI+DTPSI                                                    194
      A1=1.+H*(VB-VA+(AB-AA)*SIN(TPSI))/DZ                               195
      B1=H*(VC-VA+(AC-AA)*SIN(TPSI))/DR                                  196
      C1=-(ZZ-Z0+H*(VA+AA*SIN(TPSI)))                                    197
      A2=H*(UB-UA+(AB-AA)*COS(TPSI))/DZ                                  198
      B2=1.+H*(UC-UA+(AC-AA)*COS(TPSI))/DR                               199
      C2=-(RR-R0+H*(UA+AA*COS(TPSI)))                                    200
      DET=A1*B2-A2*B1                                                    201
      DELX=(B2*C1-B1*C2)/DET                                             202
      DELY=(A1*C2-A2*C1)/DET                                             203
C                                                                        204
C         TEST FOR SAME REGION                                           205
C                                                                        206
      XB1=XB+DELX                                                        207
      YB1=YB+DELY                                                        208
      M=COMP(XB,YB,XB1,YB1)                                              209
      IF (LM.EQ.2) GO TO 6700                                            210
```

```
      IF (M.EQ.1) GO TO 6203                                                    211
      GO TO 6800                                                                212
6700  CONTINUE                                                                  213
      IF (M.NE.1) GO TO 6203                                                    214
6800  CONTINUE                                                                  215
      MM=MM+1                                                                   216
      IF (MM.LE.360/NOM) GO TO 6201                                             217
612   WRITE (3,613) ITIL                                                        218
613   FORMAT (44HOFAILED TO FIND 2 POINTS IN THE SAME REGION 21H IN SUBR        219
     1. NRIT2 AFTER,I4,6H TRIES)                                                220
      WRITE (3,614) ZO,RO                                                       221
614   FORMAT (1X5HZO  =,E15.8,4X5HRO  =,E15.8)                                  222
      WRITE (3,6144) LM                                                         223
      WRITE (3,6145) M                                                          224
      WRITE (3,6146) KODE                                                       225
      WRITE (3,7002) XB,YB,XB1,YB1                                              226
6144  FORMAT (4H LM=,I4)                                                        227
6145  FORMAT (3H M=,I4)                                                         228
6146  FORMAT (6H KODE=,I4)                                                      229
      CALL EXIT                                                                 230
6203  GO TO (6204,6205),LM                                                      231
6204  ANG1=TPSI                                                                 232
      MM=0                                                                      233
      GO TO 6210                                                                234
6205  ANG2=TPSI-DTPSI                                                           235
6210  CONTINUE                                                                  236
      AL=LLL+1                                                                  237
      DO 6300 KK=1,LLL                                                          238
      ISUB=LL(KK)                                                               239
      AK=KK                                                                     240
      PSI(ISUB)=ANG1+(ANG2-ANG1)*AK/AL                                          241
6300  CONTINUE                                                                  242
      NBIC=1                                                                    243
      GO TO 619                                                                 244
C                                                                               245
C                                                                               246
```

```
6400  CONTINUE                                                          247
      IF (L1.EQ.2.OR.L1.EQ.3) GO TO 642                                 248
C     CALL NRIT2(ZI(8),RI(8),ZZ,DZ,RR,DR,EPI1,EPI2,FGOF5,ITI1,KODE)     249
      CALL NRIT2(ZI(8),RI(8),ZZ,DZ,RR,DR,EPI1,EPI2,FGOF5,ITI1,KODE,
     1 PHI,NMAX,MMAX)
      IF (KODE.EQ.0) GO TO 635                                          250
      ISUB=8                                                            251
      WRITE (3,622) ITI1,I,J,ISUB,ZO,RO,ZZ,RR,ZI(8),RI(8)               252
622   FORMAT (27HOFAILED TO FIND ZI,RI AFTER,I4,6H TRIES,3X2HI=,I4,3X2HJ  253
     1=,I4,3X5HISUB=,I4/1X3HZO=,E15.8,6X3HRO=,E15.8/1X3HZZ=,E15.8,6X3HRR  254
     1=,E15.8/1X3HZI=,E15.8,6X3HRI=,E15.8/1H1)                          255
      CALL EXIT                                                         256
C                                                                       257
C                                                                       258
  635 IF(TIME.GT.TSSS) GO TO 132
      CALL DBLTRP(ZI(8),RI(8),ANS)
       GO TO 133
  132 CALL DSURFT(ZI(8),RI(8),ANS,PHI,NMAX,MMAX)
  133 PI(8)=ANS(1)
C     PI(8)=ANS(1)
      UI(8)=ANS(2)                                                      261
      VI(8)=ANS(3)                                                      262
      RHOI(8)=ANS(4)                                                    263
      EI(8)=ANS(5)                                                      264
      AI(8)=ANS(6)                                                      265
C                                                                       266
642   DO 670 IL=1,LLL                                                   267
      NN=LL(IL)                                                         268
      M=PART(1,ZI(NN),RI(NN),ZX,RX,DELTA,NDEL)                          269
      IF (M.EQ.1) GO TO 645                                             270
      PUR(NN)=0.                                                        271
      PVR(NN)=0.                                                        272
      GO TO 648                                                         273
  645 IF(TIME.GT.TSSS) GO TO 134
      CALL DBLTRP(ZX,RX,ANS)
      GO TO 135
```

```
  134 CALL DSURFT(ZX,RX,ANS,PHI,NMAX,MMAX)
  135 DEN=RX-RI(NN)
C     DEN=RX-RI(NN)
      PUR(NN)=(ANS(2)-UI(NN))/DEN                                          276
      PVR(NN)=(ANS(3)-VI(NN))/DEN                                          277
648   M=PART(2,ZI(NN),RI(NN),ZX,RX,DELTA,NDEL)                             278
      IF (M.EQ.1) GO TO 650                                                279
      PUZ(NN)=0.                                                           280
      PVZ(NN)=0.                                                           281
      GO TO 655                                                            282
  650 IF(TIME.GT.TSSS) GO TO 136
      CALL DBLTRP(ZX,RX,ANS)
      GO TO 137
  136 CALL DSURFT(ZX,RX,ANS,PHI,NMAX,MMAX)
  137 DEN=ZX-ZI(NN)
      PUZ(NN)=(ANS(2)-UI(NN))/DEN                                          285
      PVZ(NN)=(ANS(3)-VI(NN))/DEN                                          286
655   S(NN)=SPSI(NN)**2*PUR(NN)-CPSI(NN)*SPSI(NN)*(PVR(NN)+PUZ(NN))+CPSI   287
     1(NN)**2*PVZ(NN)                                                      288
C                                                                          289
      IF (ABS(RI(NN)).GT.EPS) GO TO 660                                    290
      CON=PUR(NN)                                                          291
      GO TO 662                                                            292
660   CON=UI(NN)/RI(NN)                                                    293
662   S(NN)=-RHOI(NN)*H*AI(NN)**2*(S(NN)+CON)+PI(NN)+RHOI(NN)*AI(NN)*CPS   294
     1I(NN)*UI(NN)+RHOI(NN)*AI(NN)*SPSI(NN)*VI(NN)                         295
670   CONTINUE                                                             296
1005  CONTINUE                                                             297
1002  FORMAT (4E16.8)                                                      298
      KICK=670                                                             299
      CALL DVCHK(KQ)                                                       300
      IF (KQ.EQ.1) GO TO 9980                                              301
C                                                                          302
C         COMPUTE NEW P,U,V                                                303
C                                                                          304
      GO TO (690,692,695,698),L1                                           305
```

```
690   DUO=0.                                                                  306
      U02=0.                                                                  307
      V02=(S(4)-S(6))/(RHOI(4)*AI(4)*SPSI(4)-RHOI(6)*AI(6)*SPSI(6))           308
      P02=S(4)-RHOI(4)*AI(4)*SPSI(4)*V02                                      309
      GO TO 700                                                               310
C                                                                             311
C                                                                             312
692   DP0=0.                                                                  313
      P02=0.                                                                  314
      L=LL(1)                                                                 315
      TMP1=RHOI(7)*RHOI(L)*AI(7)*AI(L)*(CPSI(7)*SPSI(L)-CPSI(L)*SPSI(7))      316
      V02=(S(L)*RHOI(7)*AI(7)*CPSI(7)-S(7)*RHOI(L)*AI(L)*CPSI(L))/TMP1        317
      U02=(S(L)-RHOI(L)*AI(L)*SPSI(L)*V02)/(RHOI(L)*AI(L)*CPSI(L))            318
      GO TO 700                                                               319
C                                                                             320
C                                                                             321
695   DP0=0.                                                                  322
      P02=0.                                                                  323
      TMP1=RHOI(7)*RHOI(6)*AI(7)*AI(6)*(CPSI(7)*SPSI(6)-SPSI(7)*CPSI(6))      324
      V02=(S(6)*RHOI(7)*AI(7)*CPSI(7)-S(7)*RHOI(6)*AI(6)*CPSI(6))/TMP1        325
      U02=(S(6)-RHOI(6)*AI(6)*SPSI(6)*V02)/(RHOI(6)*AI(6)*CPSI(6))            326
      GO TO 700                                                               327
698   CONTINUE                                                                328
      L=LL(1)                                                                 329
      TMP1=RHOI(4)*AI(4)*CPSI(4)-RHOI(L)*AI(L)*CPSI(L)                        330
      TMP2=RHOI(4)*AI(4)*SPSI(4)-RHOI(L)*AI(L)*SPSI(L)                        331
      TMP3=RHOI(6)*AI(6)*CPSI(6)-RHOI(L)*AI(L)*CPSI(L)                        332
      TMP4=RHOI(6)*AI(6)*SPSI(6)-RHOI(L)*AI(L)*SPSI(L)                        333
      V02=((S(4)-S(L))*TMP3-(S(6)-S(L))*TMP1)/(TMP3*TMP2-TMP1*TMP4)           334
      U02=(S(4)-S(L)-TMP2*V02)/TMP1                                           335
      P02=S(6)-RHOI(6)*AI(6)*CPSI(6)*U02-RHOI(6)*AI(6)*SPSI(6)*V02            336
700   KICK=700                                                                337
      CALL DVCHK(KQ)                                                          338
      IF (KQ.EQ.1) GO TO 9980                                                 339
C                                                                             340
C                                                                             341
```

```
C          ITERATE FOR RHOO,EO                                                    342
C                                                                                 343
705    ITI33=ITI3                                                                 344
       ITI44=ITI4                                                                 345
       KM=1                                                                       346
708    CONTINUE                                                                   347
       CALL EQOSI(PO2,PGRHO,PGE,BIGG,CHECK,KRTT,AO2,EO2,RHOO2,KM,EPS)             348
       IF (KRTT.EQ.1) GO TO 871                                                   349
725    T1=RHOO-RHOI(8)                                                            350
       T2=PI(8)/RHOI(8)**2                                                        351
       BIGH=EI(8)+T2*T1-EO                                                        352
       PHE=-1.                                                                    353
       PHRHO=T2                                                                   354
       KICK=725                                                                   355
       CALL DVCHK(KQ)                                                             356
       IF (KQ.EQ.1) GO TO 9980                                                    357
C                                                                                 358
C          COMPUTE NEW EO,RHOO                                                    359
C                                                                                 360
       DOWN=PGE*PHRHO-PGRHO*PHE                                                   361
       DEO=(-BIGG*PHRHO+BIGH*PGRHO)/DOWN                                          362
       DRHOO=(-BIGH*PGE+BIGG*PHE)/DOWN                                            363
       EO2=EO+DEO                                                                 364
       RHOO2=RHOO+DRHOO                                                           365
C                                                                                 366
C          CHECK EO2,RHOO2 FOR CONVERGENCE                                        367
       KICK=726                                                                   368
       CALL DVCHK(KQ)                                                             369
       IF (KQ.EQ.1) GO TO 9980                                                    370
C                                                                                 371
       IF (ABS(DEO/EO).LT.EPI7) GO TO 726                                         372
       IF (ABS(DEO).GT..01*EPI7) GO TO 730                                        373
726    IF (ABS(DRHOO/RHOO).LE.EPI6) GO TO 740                                     374
       IF (ABS(DRHOO).LT.EPI6) GO TO 740                                          375
730    ITI33=ITI33-1                                                              376
       IF (ITI33.NE.0) GO TO 735                                                  377
```

```
      WRITE (6,732) ITI3,I,J,Z0,R0,P0,U0,V0,RHO0,E0,P02,U02,V02,RHO02,E0      378
     12                                                                      379
732   FORMAT (33HOE0,RHO0 FAILED TO CONVERGE AFTER,I4,6H TRIES/1X2HI=,I4     380
     1,4X2HJ=,I4,4X2HZ=,E15.8,4X2HR=,E15.8/5X2HP018X2HU018X2HV018X4HRHO0    381
     116X2HE0/5X3HP0217X3HU0217X3HV0217X5HRHO0215X3HE02//(5E20.8))          382
      WRITE (3,1)                                                           383
      CALL EXIT                                                             384
735   E0=E02                                                                385
      RHO0=RHO02                                                            386
      KM=0                                                                  387
      GO TO 708                                                             388
C                                                                           389
C          CHECK FOR PROPER EQUATIONS                                       390
C                                                                           391
740   CONTINUE                                                              392
      PGE=-PGE                                                              393
      PGRHO=-PGRHO                                                          394
      IF (RHO02.GE.RHOSTR) GO TO 750                                        395
      IF (E02.LT.EPRS) GO TO 750                                            396
742   CHECK2=0                                                              397
      GO TO 752                                                             398
750   CHECK2=1                                                              399
752   A02=SQRT(+PGRHO+P02*PGE/RHO02**2)                                     400
      IF (CHECK.EQ.CHECK2) GO TO 870                                        401
      ITI44=ITI44-1                                                         402
      IF (ITI44.NE.0) GO TO 770                                             403
      WRITE (3,755) ITI4,I,J,Z0,R0,P0,U0,V0,RHO0,E0,A0,P02,U02,V02,RHO02    404
     1,E02,A02                                                              405
755   FORMAT (38HOFAILED TO USE CORRECT EQUATIONS AFTER,I4,6H TRIES/1X2H     406
     1I=,I4,4X2HJ=,I4,4X2HZ=,E15.8,4X2HR=,E15.8/5X2HP018X2HU018X2HV018X4    407
     1HRHO016X2HE018X2HA0/5X3HP0217X3HU0217X3HV0217X5HRHO0215X3HE0217X3H    408
     1A02//(6E20.8))                                                        409
      WRITE (3,1)                                                           410
      CALL EXIT                                                             411
770   KICK=770                                                              412
      CALL DVCHK(KQ)                                                        413
```

```
      IF (KQ.EQ.1) GO TO 9980                                                414
      ITI33=ITI3                                                             415
      GO TO 735                                                              416
C                                                                            417
C          ALL VALUES HAVE CONVERGED FOR 1 INTERIOR POINT                    418
C                                                                            419
871   KRTT=0                                                                 420
870   XMESH2(I,J,1)=PO2                                                      421
      XMESH2(I,J,2)=UO2                                                      422
      XMESH2(I,J,3)=VO2                                                      423
      XMESH2(I,J,4)=RHOO2                                                    424
      XMESH2(I,J,5)=EO2                                                      425
      XMESH2(I,J,6)=AO2                                                      426
      KICK=900                                                               427
      CALL DVCHK(KQ)                                                         428
      IF (KQ.EQ.1) GO TO 9980                                                429
900   CONTINUE                                                               430
905   CONTINUE                                                               431
      IF (ITS3.EQ.1) GO TO 950                                               432
      CALL ITRP                                                              433
      RETURN                                                                 434
950   CONTINUE                                                               435
9980  WRITE (3,9985) KICK                                                    436
      WRITE (3,614) ZO,RO                                                    437
9985  FORMAT (32HODIVIDE CHECK NEAR STATEMENT NO.,I5,15H IN SUBR. INTER/     438
     11H1)                                                                   439
      CALL EXIT                                                              440
      RETURN                                                                 441
      DEBUG SUBCHK
      END                                                                    442
      SUBROUTINE EQOS1(PRHO,PPP,PVV,PEE,TEE,TRHO,KICK)                         1
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPS       2
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LEN       3
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS              4
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2       5
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(      6
```

```
     115,4),RPART(15,2)                                                      7
C                                                                            8
C                                                                            9
      COMMON ZO,RO,PO,UO,VO,LO,MO,RHOO,EO,AO,UBARO,VBARO                    10
C                                                                           11
      COMMON NP,NT,NR,NI,NDEL,ISUB                                          12
C                                                                           13
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                       14
      COMMON DIRCOS                                                         15
      COMMON TIME                                                           16
      COMMON IRARF                                                          17
      COMMON KSTOP                                                          18
      COMMON TPSI                                                           19
      COMMON KKK                                                            20
C                                                                           21
C                                                                           22
      REAL LO,MO,LENGTH,MU,KO                                               23
      RHO=RHOSTR                                                            24
      E=VP**2/8.                                                            25
      DO 100 M=1,100                                                        26
      ETA=RHO/RHOSTR                                                        27
      MU=ETA-1.                                                             28
      G=-RHOSTR*(VP/2.)**2+((APR+BPR/(E/(ESTAR*ETA**2)+1.))*E*RHO+BIGAPR    29
     1*MU+BIGBPR*MU**2)*(1.-RHOSTR/RHO)                                     30
      DERIVG=((APR+BPR/(E/(ESTAR*ETA**2)+1.))*E+BIGAPR/RHOSTR+2.*BIGBPR*    31
     1MU/RHOSTR+2.*E**2*BPR/(ESTAR*ETA**2*(E/(ESTAR*ETA**2)+1.)**2))*(1.    32
     1-RHOSTR/RHO)+((APR+BPR/(E/(ESTAR*ETA**2)+1.))*E*RHO+BIGAPR*MU+BIGB    33
     1PR*MU**2)*RHOSTR/RHO**2                                               34
      DLTRHO=-G/DERIVG                                                      35
      RHO=RHO+DLTRHO                                                        36
C     IF(ABS(DLTRHO).LT.1.E-07) GO TO 101                                   37
      IF (ABS(DLTRHO).LT.1.E-06) GO TO 101                                  38
100   CONTINUE                                                              39
      KICK=2200                                                             40
      GO TO 9980                                                            41
101   CONTINUE                                                              42
```

```
      PRHO=RHO                                                          43
      TRHO=RHO                                                          44
      PEE=E                                                             45
      TEE=E                                                             46
      PVV=VP/2.                                                         47
      PPP=(APR+BPR/(E/(ESTAR*ETA**2)+1.))*E*RHO+BIGAPR*MU+BIGBPR*MU**2  48
9980  RETURN                                                            49
      DEBUG SUBCHK
      END                                                               50
      SUBROUTINE EQOS2(PPP,PRHO,PEE)                                     1
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPS 2
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LEN 3
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS        4
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2  5
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(  6
     115,4),RPART(15,2)                                                  7
C                                                                        8
C                                                                        9
      COMMON ZO,RO,PO,UO,VO,LO,MO,RHOO,EO,AO,UBARO,VBARO                10
C                                                                       11
      COMMON NP,NT,NR,NI,NDEL,ISUB                                      12
C                                                                       13
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                   14
      COMMON DIRCOS                                                     15
      COMMON TIME                                                       16
      COMMON IRARF                                                      17
      COMMON KSTOP                                                      18
C                                                                       19
      COMMON TPSI                                                       20
      COMMON KKK                                                        21
C                                                                       22
      REAL LO,MO,LENGTH,MU,KO                                           23
      P=PPP                                                             24
      RHO=PRHO                                                          25
      E=PEE                                                             26
      ETA=RHO/RHOSTR                                                    27
```

```
      MU=ETA-1.                                                          28
      EE=E/(ESTAR*ETA**2)+1.                                             29
      PGRHO=E*(APR+BPR/EE)+BIGAPR/RHOSTR+(2.*BIGBPR*MU)/RHOSTR+(2.*E**2*  30
     1BPR)/(ESTAR*ETA**2*EE**2)                                          31
      PGE=(APR+BPR/EE)*RHO-(E*BPR*RHO)/(ESTAR*ETA**2*EE**2)              32
      AR=SQRT(PGRHO+PGE*P/RHO**2)                                        33
      RETURN                                                             34
      DEBUG SUBCHK
      END                                                                35
      SUBROUTINE EQOS3(RHO,AA,E,P)                                        1
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPS  2
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LEN  3
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS         4
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2  5
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(  6
     115,4),RPART(15,2)                                                   7
C                                                                         8
C                                                                         9
      COMMON ZO,RO,PO,UO,VO,LO,MO,RHOO,EO,AO,UBARO,VBARO                 10
C                                                                        11
      COMMON NP,NT,NR,NI,NDEL,ISUB                                       12
C                                                                        13
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                    14
      COMMON DIRCOS                                                      15
      COMMON TIME                                                        16
      COMMON IRARF                                                       17
      COMMON KSTOP                                                       18
      COMMON TPSI                                                        19
      COMMON KKK                                                         20
C                                                                        21
C                                                                        22
      REAL LO,MO,LENGTH,MU,KO                                            23
70    ETA=RHO/RHOSTR                                                     24
      MU=ETA-1.                                                          25
      EE=E/(ESTAR*ETA**2)+1.                                             26
      IF (RHO.GT.RHOSTR) GO TO 72                                        27
```

```
      IF (E.GE.EPRS) GO TO 74                                                28
72    PGRHO=E*(APR+BPR/EE)+BIGAPR/RHOSTR+(2.*BIGBPR*MU)/RHOSTR+(2.*E**2*     29
     1BPR)/(ESTAR*ETA**2*EE**2)                                              30
      PGE=(APR+BPR/EE)*RHO-(E*BPR*RHO)/(ESTAR*ETA**2*EE**2)                  31
      GO TO 75                                                               32
74    C1=RHOSTR/RHO-1.                                                       33
      C2=EXP(-BETA*C1)                                                       34
      C3=EXP(-ALPHA*C1**2)                                                   35
      T1=(BPR*E*RHO)/EE+BIGAPR*MU*C2                                         36
      T2=2.*ALPHA*C1*(RHOSTR/(RHO**2))                                       37
      T3=BPR*E/EE                                                            38
      T4=(2.*E)/(ESTAR*ETA**2*EE)                                            39
      T4=T3*T4                                                               40
      T5=(BIGAPR*C2)/RHOSTR                                                  41
      T6=(BIGAPR*MU*BETA*RHOSTR*C2)/(RHO**2)                                 42
      PGRHO=APR*E+C3*(T1*T2+T3+T4+T5+T6)                                     43
      T7=(BPR*RHO)/EE                                                        44
      PGE=APR*RHO+C3*(T7-T7*(E/(ESTAR*ETA**2*EE)))                           45
75    AA=SQRT(PGRHO+PGE*P/RHO**2)                                            46
      RETURN                                                                 47
      DEBUG SUBCHK
      END                                                                    48
      SUBROUTINE EQOSS(PFR,PFE)                                               1
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPS      2
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LEN      3
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS             4
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2      5
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(     6
     115,4),RPART(15,2)                                                       7
C                                                                             8
C                                                                             9
      COMMON ZO,RO,PO,UO,VO,LO,MO,RHOO,EO,AO,UBARO,VBARO                     10
C                                                                            11
      COMMON NP,NT,NR,NI,NDEL,ISUB                                           12
C                                                                            13
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                        14
```

```
      COMMON DIRCOS                                                     15
      COMMON TIME                                                       16
      COMMON IRARF                                                      17
      COMMON KSTOP                                                      18
      COMMON TPSI                                                       19
      COMMON KKK                                                        20
C                                                                       21
C                                                                       22
      REAL LO,MO,LENGTH,MU,KO                                           23
250   ETA=RHOO/RHOSTR                                                   24
      MU=ETA-1.                                                         25
C                                                                       26
C                                                                       27
      EPP=EO/(ESTAR*ETA**2)+1.                                          28
      TMP=APR+BPR/EPP                                                   29
      TMP1=BPR/(ESTAR*ETA**2*EPP**2)                                    30
      PFR=TMP*EO+(BIGAPR+2.*BIGBPR*MU)/RHOSTR+2.*EO**2*TMP1             31
      PFE=TMP*RHOO-EO*RHOO*TMP1                                         32
      RETURN                                                            33
      DEBUG SUBCHK
      END                                                               34
      SUBROUTINE EQOSP(RHOO2,EO2,PO2)                                    1
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPS 2
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LEN 3
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS        4
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2  5
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(  6
     115,4),RPART(15,2)                                                  7
C                                                                        8
C                                                                        9
      COMMON ZO,RO,PO,UO,VO,LO,MO,RHOO,EO,AO,UBARO,VBARO                10
C                                                                       11
      COMMON NP,NT,NR,NI,NDEL,ISUB                                      12
C                                                                       13
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                   14
      COMMON DIRCOS                                                     15
```

```
      COMMON TIME                                                       16
      COMMON IRARF                                                      17
      COMMON KSTOP                                                      18
      COMMON TPSI                                                       19
      COMMON KKK                                                        20
C                                                                       21
C                                                                       22
      REAL LO,MO,LENGTH,MU,KO                                           23
      ETA=RHO02/RHOSTR                                                  24
      MU=ETA-1.                                                         25
      PO2=EO2*RHO02*(APR+(BPR*ESTAR*ETA**2)/(EO2+ESTAR*ETA**2))+BIGAPR*M 26
     1U+BIGBPR*MU**2                                                    27
      RETURN                                                            28
      DEBUG SUBCHK
      END                                                               29
      SUBROUTINE EQOSI(PO2,PGRHO,PGE,BIGG,CHECK,KRTT,AO2,EO2,RHO02,KM,EP 1
     1S)                                                                2
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPS 3
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LEN 4
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS        5
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2 6
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2( 7
     115,4),RPART(15,2)                                                 8
C                                                                       9
C                                                                       10
      COMMON ZO,RO,PO,UO,VO,LO,MO,RHOO,EO,AO,UBARO,VBARO                11
C                                                                       12
      COMMON NP,NT,NR,NI,NDEL,ISUB                                      13
C                                                                       14
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                   15
      COMMON DIRCOS                                                     16
      COMMON TIME                                                       17
      COMMON IRARF                                                      18
      COMMON KSTOP                                                      19
      COMMON TPSI                                                       20
      COMMON KKK                                                        21
```

```
C                                                                       22
C                                                                       23
      REAL LO,MO,LENGTH,MU,KO                                           24
      INTEGER CHECK,CHECK2                                              25
      IF (KM.EQ.0) GO TO 708                                            26
      RHOO=RHOSTR                                                       27
      TMP1=(APR+BPR)*RHOSTR-PO2/ESTAR                                   28
      TMP2=SQRT(TMP1**2+4.*PO2*APR*RHOSTR/ESTAR)                        29
      EO=(-TMP1+TMP2)/(2.*APR*RHOSTR/ESTAR)                             30
      IF (PO2.GT.EPS) GO TO 708                                         31
      PO2=0.                                                            32
      EO2=0.                                                            33
      RHOO2=RHOSTR                                                      34
      AO2=SQRT(BIGAPR/RHOSTR)                                           35
      KRTT=1                                                            36
      GO TO 870                                                         37
708   KRTT=0                                                            38
      ETA=RHOO/RHOSTR                                                   39
      MU=ETA-1.                                                         40
      EE=EO/(ESTAR*ETA**2)+1.                                           41
      IF (RHOO.GT.RHOSTR) GO TO 720                                     42
      IF (EO.LT.EPRS) GO TO 720                                         43
715   C1=RHOSTR/RHOO-1.                                                 44
      BEC1=BETA*C1                                                      45
      IF (BEC1.LT.10.E10) GO TO 716                                     46
      C2=0.0                                                            47
      GO TO 717                                                         48
716   C2=EXP(-BEC1)                                                     49
717   C3AL=ALPHA*C1**2                                                  50
      IF (C3AL.LT.10.E12) GO TO 718                                     51
      C3=0.0                                                            52
      GO TO 719                                                         53
718   C3=EXP(-C3AL)                                                     54
719   CONTINUE                                                          55
      T1=(BPR*EO*RHOO)/EE+BIGAPR*MU*C2                                  56
      T2=2.*ALPHA*C1*(RHOSTR/(RHOO**2))                                 57
```

```
      T3=BPR*E0/EE                                                      58
      T4=(2.*E0)/(ESTAR*ETA**2*EE)                                      59
      T4=T3*T4                                                          60
      T5=(BIGAPR*C2)/RHOSTR                                             61
      T6=(BIGAPR*MU*BETA*RHOSTR*C2)/(RHO0**2)                           62
      T7=(BPR*RHO0)/EE                                                  63
      PGRHO=APR*E0+C3*(T1*T2+T3+T4+T5+T6)                               64
      PGE=APR*RHO0+C3*(T7-T7*(E0/(ESTAR*ETA**2*EE)))                    65
      BIGG=P02-APR*E0*RHO0-T1*C3                                        66
      CHECK=0                                                           67
      GO TO 725                                                         68
720   T1=APR+BPR/EE                                                     69
      T2=(BPR*E0)/(ESTAR*ETA**2*EE**2)                                  70
      BIGG=P02-T1*E0*RHO0-BIGAPR*MU-BIGBPR*MU**2                        71
      PGRHO=T1*E0+BIGAPR/RHOSTR+2.*BIGBPR*MU/RHOSTR+T2*2.*E0            72
      PGRHO=-PGRHO                                                      73
      PGE=T1*RHO0-RHO0*T2                                               74
      PGE=-PGE                                                          75
      CHECK=1                                                           76
725   CONTINUE                                                          77
870   CONTINUE                                                          78
      RETURN                                                            79
      DEBUG SUBCHK
      END                                                               80
      SUBROUTINE SOUT2                                                   1
C                                                                        2
C         PRINTS 6 LINES OF DISCONTINUITY AT TO                          3
C                                                                        4
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPS  5
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LEN  6
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS         7
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2  8
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(  9
     115,4),RPART(15,2)                                                  10
C                                                                        11
C                                                                        12
```

```
      COMMON ZO,RO,PO,UO,VO,LO,MO,RHOO,EO,AO,UBARO,VBARO                13
C                                                                       14
      COMMON NP,NT,NR,NI,NDEL,ISUB                                      15
C                                                                       16
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                   17
      COMMON DIRCOS                                                     18
      COMMON TIME                                                       19
      COMMON IRARF                                                      20
      COMMON KSTOP                                                      21
      COMMON TPSI                                                       22
      COMMON KKK                                                        23
      REAL LO,MO,LENGTH,MU,KO                                           24
C                                                                       25
4     FORMAT (///30H CURVES OF DISCONTINUITY AT TO//40X20H--PROJECTILE S 26
     1HOCK--//)                                                         27
6     FORMAT (//35X29H--PROJECTILE PARTICLE CURVE--//)                  28
8     FORMAT (//42X16H--TARGET SHOCK--//)                               29
10    FORMAT (//38X25H--TARGET PARTICLE CURVE--//)                      30
12    FORMAT (//42X15H--RAREFACTION--//)                                31
14    FORMAT (//35X30H--RAREFACTION PARTICLE CURVE--//)                 32
16    FORMAT (7X1HZ19X1HR19X1HP19X1HU19X1HV/7X3HRHO17X1HE19X1HA19X1HL19X 33
     11HM//)                                                            34
18    FORMAT (7X1HZ19X1HR//)                                            35
20    FORMAT (7X1HZ19X1HR19X1HL19X1HM//)                                36
30    FORMAT (5E20.8/5E20.8//)                                          37
35    FORMAT (2E20.8)                                                   38
38    FORMAT (4E20.8)                                                   39
39    FORMAT (//35X16H--FREE SURFACE--//)                               40
40    FORMAT (1H1)                                                      41
      WRITE (3,4)                                                       42
      WRITE (3,16)                                                      43
      WRITE (3,30) ((TAB2(I,J,1),J=1,10),I=1,NP)                        44
      WRITE (3,6)                                                       45
      WRITE (3,18)                                                      46
      WRITE (3,35) ((SPART(I,J,1),J=1,2),I=1,NP)                        47
      WRITE (3,8)                                                       48
```

```
      WRITE (3,16)                                                      49
      WRITE (3,30) ((TAB2(I,J,2),J=1,10),I=1,NT)                        50
      WRITE (3,10)                                                      51
      WRITE (3,18)                                                      52
      WRITE (3,35) ((SPART(I,J,2),J=1,2),I=1,NT)                        53
      WRITE (3,12)                                                      54
      WRITE (3,20)                                                      55
      WRITE (3,38) ((RARF2(I,J),J=1,4),I=1,NR)                          56
      WRITE (3,14)                                                      57
      WRITE (3,39)                                                      58
      WRITE (3,35) ((SURF2(I,J),J=1,2),I=1,NP)                          59
      WRITE (3,40)                                                      60
      RETURN                                                            61
      DEBUG SUBCHK
      END                                                               62
      SUBROUTINE SOUT                                                    1
C                                                                        2
C       PRINTS 4 LINES OF DISCONTINUITY AT TO-H                          3
C                                                                        4
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPS 5
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LEN 6
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS         7
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2  8
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(  9
     115,4),RPART(15,2)                                                  10
C                                                                        11
C                                                                        12
      COMMON ZO,RO,PO,UO,VO,LO,MO,RHOO,EO,AO,UBARO,VBARO                 13
C                                                                        14
      COMMON NP,NT,NR,NI,NDEL,ISUB                                       15
C                                                                        16
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                    17
      COMMON DIRCOS                                                      18
      COMMON TIME                                                        19
      COMMON IRARF                                                       20
      COMMON KSTOP                                                       21
```

```
      COMMON TPSI                                                       22
      COMMON KKK                                                        23
      REAL LO,MO,LENGTH,MU,KO                                           24
C                                                                       25
4     FORMAT (///32H CURVES OF DISCONTINUITY AT TO-H//40X20H--PROJECTILE 26
     1 SHOCK--//)                                                       27
6     FORMAT (//42X16H--TARGET SHOCK--//)                               28
8     FORMAT (//42X15H--RAREFACTION--//)                                29
10    FORMAT (7X1HZ19X1HR19X1HP19X1HU19X1HV/7X3HRHO17X1HE19X1HA19X1HL19X 30
     11HM//)                                                            31
15    FORMAT (5E20.8/5E20.8//)                                          32
18    FORMAT (1H1)                                                      33
21    FORMAT (//35X16H--FREE SURFACE--//)                               34
25    FORMAT (2E20.8)                                                   35
      WRITE (3,4)                                                       36
      WRITE (3,10)                                                      37
      WRITE (3,15) ((TAB(I,J,1),J=1,10),I=1,NP)                         38
      WRITE (3,6)                                                       39
      WRITE (3,10)                                                      40
      WRITE (3,15) ((TAB(I,J,2),J=1,10),I=1,NT)                         41
      WRITE (3,8)                                                       42
      WRITE (3,10)                                                      43
      WRITE (3,15) ((RARF(I,J),J=1,10),I=1,NR)                          44
      WRITE (3,21)                                                      45
      WRITE (3,25) ((SURF(I,J),J=1,2),I=1,NP)                           46
      WRITE (3,18)                                                      47
      RETURN                                                            48
      DEBUG SUBCHK
      END                                                               49
      SUBROUTINE PRINT(BL,ZTAB,RTAB,KK,PHI,NMAX,MMAX)
C     SUBROUTINE PRINT(BL,ZTAB,RTAB,KK)
      DOUBLE PRECISION XS(20),RU(20),ZU(20,20,6),UU(20),WW(20),
     1ZCOMP(20,20,6),PHI(20,20,6),CETA(20,20),DENC,POLY
C          PRINTS INTERIOR REGION                                        2
C                                                                        3
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPS 4
```

```
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LEN      5
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS             6
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2       7
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(      8
     115,4),RPART(15,2)                                                       9
C                                                                            10
C                                                                            11
      COMMON Z0,R0,P0,U0,V0,L0,M0,RHO0,E0,A0,UBAR0,VBAR0                     12
C                                                                            13
      COMMON NP,NT,NR,NI,NDEL,ISUB                                           14
C                                                                            15
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                        16
      COMMON DIRCOS                                                          17
      COMMON TIME                                                            18
      COMMON IRARF                                                           19
      COMMON KSTOP                                                           20
      COMMON TPSI                                                            21
      COMMON KKK                                                             22
      REAL L0,M0,LENGTH,MU,K0                                                23
C                                                                            24
      DIMENSION BL(20,20,6),ZTAB(20),RTAB(20)                                25
C                                                                            26
      TSSS=1.1
      M=0
      N=0
      DO 15 I=1,20                                                           27
      DO 15 J=1,20                                                           28
      IF (ABS(BL(I,J,1))+ABS(BL(I,J,2))+ABS(BL(I,J,3)))  20,15,20            29
15    CONTINUE                                                               30
      WRITE (3,18)                                                           31
18    FORMAT (15H1TABLES ALL = 0/1H1)                                        32
      CALL EXIT                                                              33
20    I1=I                                                                   34
      DO 30 I=I1,20                                                          35
      IF (ABS(BL(I,1,1))+ABS(BL(I,1,2))+ABS(BL(I,1,3)))  30,22,30            36
22    DO 25 J=1,20                                                           37
```

```
      IF (ABS(BL(I,J,1))+ABS(BL(I,J,2))+ABS(BL(I,J,3)))  30,25,30
25    CONTINUE
      I2=I-1
      GO TO 35
30    CONTINUE
      I2=20
35    DO 45 J=1,20
      IF (ABS(BL(I1,J,1))+ABS(BL(I1,J,2))+ABS(BL(I1,J,3)))  45,37,45
37    DO 40 I=I1,I2
      IF (ABS(BL(I,J,1))+ABS(BL(I,J,2))+ABS(BL(I,J,3)))  45,40,45
40    CONTINUE
      J2=J-1
      GO TO 50
45    CONTINUE
      J2=20
50    J1=1
C
C         PRINT TABLE
C
      GO TO (52,56),KK
52    WRITE (3,53)
53    FORMAT (//24HOINTERIOR REGION AT TO-H///)
      GO TO 62
56    WRITE (3,57)
57    FORMAT (//22HOINTERIOR REGION AT TO///)
62    DO 70 I=I1,I2
      WRITE (3,64) ZTAB(I)
64    FORMAT (///7HOZTAB =,F10.4//7X1HR9X1HP17X1HU17X1HV17X3HRHO15X1HE17
     1X1HA//)
      M=M+1
      XS(M)=ZTAB(I)
      UU(M)=1.0
      DO 69 J=J1,J2
      WRITE (3,68) RTAB(J),(BL(I,J,K),K=1,6)
      N=N+1
      RU(N)=RTAB(J)
```

```
      WW(N)=1.0
      DO 156 IK=1,6
C     ZU(M,N)=BL(I,J,4)
C     ZZ(M,N)=BL(I,J,1)
  156 ZU(M,N,IK)=BL(I,J,IK)
68    FORMAT (F12.4,6E18.8)                                          69
69    CONTINUE                                                       70
      MMAX=N
      N=0
70    CONTINUE                                                       71
      NMAX=M
C     IF(TIME.LT.TSSS) GO TO 80
  333 IMAX=I2+1-I1
      JMAX=J2+1-J1
      CALL SURFIT(XS,UU,RU,WW,ZU,NMAX,MMAX,IMAX,JMAX,CETA,PHI,ZCOMP,
     1SQD,SQDC,SDC,DFC)
      WRITE(3,100)
  100 FORMAT('  *** ')
      DO 88 IK=1,6
      DO 48 I=1,IMAX
      DO 48 J=1,JMAX
      DENC=0.D0
      DO 43 IS=1,NMAX
      K=NMAX-IS+1
      POLY=PHI(K,MMAX,IK)
      IF(MMAX-1)43,43,46
   46 DO 42 IT=2,MMAX
      L=MMAX-IT+1
   42 POLY=POLY*RU(J)+PHI(K,L,IK)
   43 DENC=DENC*XS(I)+POLY
      ZCOMP(I,J,IK)=DENC
   48 CONTINUE
   88 CONTINUE
80    WRITE (3,82)                                                   72
82    FORMAT (1H1)                                                   73
      RETURN                                                         74
```

```
      DEBUG SUBCHK
      END                                                               75
      SUBROUTINE NRIT2(X,Y,XO,DX,YO,DY,EX,EY,FGOF,IT,KODE,
     1PHI,NMAX,MMAX)
C     SUBROUTINE NRIT2(X,Y,XO,DX,YO,DY,EX,EY,FGOF,IT,KODE)               1
C                                                                        2
C         NEWTON-RAPHSON METHOD FOR SOLUTION OF                          3
C         TWO NON LINEAR EQUATIONS IN TWO UNKNOWNS                       4
C                                                                        5
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPS  6
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LEN  7
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS         8
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2  9
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(  10
     115,4),RPART(15,2)                                                  11
C                                                                        12
C                                                                        13
      COMMON ZO,RO,PO,UO,VO,LO,MO,RHOO,EO,AO,UBARO,VBARO                 14
C                                                                        15
      COMMON NP,NT,NR,NI,NDEL,ISUB                                       16
C                                                                        17
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                    18
      COMMON DIRCOS                                                      19
      COMMON TIME                                                        20
      COMMON IRARF                                                       21
      COMMON KSTOP                                                       22
      COMMON TPSI                                                        23
      COMMON KKK                                                         24
      REAL LO,MO,LENGTH,MU,KO                                            25
      DOUBLE PRECISION PHI(20,20,6)
C                                                                        26
      TSSS=1.1
      XB=XO                                                              27
      YB=YO                                                              28
      DXX=DX                                                             29
      DYY=DY                                                             30
```

```
      DELX1=0                                                          31
      DELY1=0                                                          32
      KODE=0                                                           33
1     CONTINUE                                                         34
      DO 50 I=1,IT                                                     35
      KK=0                                                             36
      XX=XB+DXX                                                        37
      YY=YB+DYY                                                        38
C                                                                      39
C                                                                      40
      IF(TIME.LT.TSSS) GO TO 266
      CALL FGOF(XB,YY,F2,G2,PHI,NMAX,MMAX)
      CALL FGOF(XX,YB,F1,G1,PHI,NMAX,MMAX)
      CALL FGOF(XB,YB,F0,G0,PHI,NMAX,MMAX)
      GO TO 267
  266 CALL FGOF(XB,YY,F2,G2)
      CALL FGOF(XX,YB,F1,G1)
      CALL FGOF(XB,YB,F0,G0)
  267 A1=(F1-F0)/DXX
      B1=(F2-F0)/DYY                                                   45
      C1=-F0                                                           46
      A2=(G1-G0)/DXX                                                   47
      B2=(G2-G0)/DYY                                                   48
      C2=-G0                                                           49
      DET=A1*B2-A2*B1                                                  50
      IF (DET.EQ.0.) GO TO 920                                         51
      DELX=(B2*C1-B1*C2)/DET                                           52
      DELY=(A1*C2-A2*C1)/DET                                           53
      IF (ABS(DELX).GT..001) GO TO 8                                   54
      DELX=0.                                                          55
8     IF (ABS(DELY).GT..001) GO TO 9                                   56
      DELY=0.                                                          57
9     CONTINUE                                                         58
      SDEL=ABS(DELX+DELX1)+ABS(DELY+DELY1)                             59
      DELX1=DELX                                                       60
      DELY1=DELY                                                       61
```

```
C                                                                       62
C          TEST FOR SAME REGION                                         63
C                                                                       64
      DO 10 J=1,IT                                                      65
      XB1=XB+DELX                                                       66
      YB1=YB+DELY                                                       67
      IF (YB1.LE.0.)YB1=0.                                              68
      M=COMP(XB,YB,XB1,YB1)                                             69
      IF (M.EQ.1) GO TO 15                                              70
11    CONTINUE                                                          71
      KK=1                                                              72
      DELX=.5*DELX                                                      73
      DELY=.5*DELY                                                      74
10    CONTINUE                                                          75
      GO TO 930                                                         76
15    IF (ABS(XB-XB1).GT.EX) GO TO 45                                   77
      IF (ABS(YB-YB1).GT.EY) GO TO 45                                   78
      IF (KK.NE.0) GO TO 45                                             79
      X=XB1                                                             80
      Y=YB1                                                             81
      KODE=0                                                            82
      RETURN                                                            83
45    CONTINUE                                                          84
      IF (KODE.NE.1) GO TO 46                                           85
      IF (SDEL.GT.EPI1) GO TO 46                                        86
      DELX=.5*DELX                                                      87
      DELY=.5*DELY                                                      88
      GO TO 9                                                           89
46    XB=XB1                                                            90
      YB=YB1                                                            91
      DEL=DELTA                                                         92
      DO 70 N=1,NDEL                                                    93
      XB2=XB+DEL                                                        94
      M=COMP(XB,YB,XB2,YB)                                              95
      IF (M.NE.1) GO TO 55                                              96
      DXX=DEL                                                           97
```

```
      GO TO 80                                                          98
55    XB2=XB-DEL                                                        99
      M=COMP(XB,YB,XB2,YB)                                             100
      IF (M.NE.1) GO TO 60                                             101
      DXX=-DEL                                                         102
      GO TO 80                                                         103
60    DEL=.5*DEL                                                       104
70    CONTINUE                                                         105
      GO TO 980                                                        106
80    DEL=DELTA                                                        107
      DO 100 N=1,NDEL                                                  108
      YB2=YB+DEL                                                       109
      M=COMP(XB,YB,XB,YB2)                                             110
      IF (M.NE.1) GO TO 85                                             111
      DYY=DEL                                                          112
      GO TO 50                                                         113
85    YB2=YB-DEL                                                       114
      M=COMP(XB,YB,XB,YB2)                                             115
      IF (M.NE.1) GO TO 90                                             116
      DYY=-DEL                                                         117
      GO TO 50                                                         118
90    DEL=.5*DEL                                                       119
100   CONTINUE                                                         120
      GO TO 990                                                        121
50    CONTINUE                                                         122
      X=XB1                                                            123
      Y=YB1                                                            124
      IF (KODE.EQ.1) RETURN                                            125
      KODE=1                                                           126
      GO TO 1                                                          127
920   WRITE (3,922) I                                                  128
922   FORMAT (46HODETERMINANT IS 0 IN SUBR. NRIT2 FOR ITERATION,I4)    129
      GO TO 950                                                        130
930   KODE=2                                                           131
      RETURN                                                           132
950   WRITE (3,952) XO,YO,XB,YB,XB1,YB1,DELX,DELY                      133
```

```
952   FORMAT (1X5HZO  =,E15.8,4X5HRO  =,E15.8/1X5HZB  =,E15.8,4X5HRB  =,      134
     1E15.8/1X5HZB1 =,E15.8,4X5HRB1 =,E15.8/1X5HDELZ=,E15.8,4X5HDELR=,E1     135
     15.8/1H1)                                                               136
      CALL EXIT                                                              137
      RETURN                                                                 138
980   KODE=3                                                                 139
      RETURN                                                                 140
990   KODE=4                                                                 141
      RETURN                                                                 142
      DEBUG SUBCHK
      END                                                                    143
      FUNCTION COMP(ZP,RP,ZP1,RP1)                                             1
C                                                                              2
C          DETERMINES IF 2 POINTS ARE IN THE SAME REGION                       3
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPS       4
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LEN       5
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS              6
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2        7
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(       8
     115,4),RPART(15,2)                                                        9
C                                                                             10
      COMMON ZO,RO,PO,UO,VO,LO,MO,RHOO,EO,AO,UBARO,VBARO                      11
C                                                                             12
      COMMON NP,NT,NR,NI,NDEL,ISUB                                            13
C                                                                             14
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                         15
      COMMON DIRCOS                                                           16
      COMMON TIME                                                             17
      COMMON IRARF                                                            18
      COMMON KSTOP                                                            19
      COMMON TPSI                                                             20
      COMMON KKK                                                              21
      REAL LO,MO,LENGTH,MU,KO                                                 22
      EPS=.0000001                                                            23
      IF (RP1.LT.0.) GO TO 80                                                 24
      IF (ZP1.GE.0.) GO TO 4                                                  25
```

```
      IF (RP1.GT.RADIUS) GO TO 80                                          26
C                                                                          27
C         FIND CONTROL CONSTANTS FOR ZP,RP                                 28
C                                                                          29
4     CONTINUE                                                             30
      IF (ITS3.EQ.1) GO TO 33                                              31
      IF (IRARF.EQ.1) GO TO 13                                             32
      M=PICK(ZP,RP,3)                                                      33
      IF (ZP.GT.RARF(M,1).AND.M.NE.1)M=M-1                                 34
      FF=RP-RARF(M+1,2)-(RARF(M,2)-RARF(M+1,2))*(ZP-RARF(M+1,1))/(RARF(M   35
     1,1)-RARF(M+1,1))                                                     36
      IF (FF.LT.0.) GO TO 11                                               37
10    NN=1                                                                 38
      GO TO 13                                                             39
11    NN=0                                                                 40
13    CONTINUE                                                             41
      DO 22 K=1,2                                                          42
      M=PICK(ZP1,RP1,K)                                                    43
      IF (RP1.LT.TAB(M,2,K).AND.M.NE.1)M=M-1                               44
5     CONTINUE                                                             45
      FF=ZP1-TAB(M+1,1,K)-(TAB(M,1,K)-TAB(M+1,1,K))*(RP1-TAB(M+1,2,K))/(   46
     1TAB(M,2,K)-TAB(M+1,2,K))                                             47
      IF (K.EQ.2) GO TO 17                                                 48
      IF (RP1.GT.RADIUS) GO TO 21                                          49
      IF (FF.LT.-EPS) GO TO 90                                             50
      GO TO 21                                                             51
17    IF (FF.GT.EPS) GO TO 100                                             52
21    IF (RP.GT.RADIUS) GO TO 22                                           53
      M=PICK(ZP1,RP1,4)                                                    54
      FF=ZP1-SURF(M+1,1)-(SURF(M,1)-SURF(M+1,1))*(RP1-SURF(M+1,2))/(SURF   55
     1(M,2)-SURF(M+1,2))                                                   56
      IF (FF)  80,22,22                                                    57
22    CONTINUE                                                             58
      IF (IRARF.EQ.1) GO TO 33                                             59
      M=PICK(ZP1,RP1,3)                                                    60
      IF (ZP1.GT.RARF(M,1).AND.M.NE.1)M=M-1                                61
```

```
      FF=RP1-RARF(M+1,2)-(RARF(M,2)-RARF(M+1,2))*(ZP1-RARF(M+1,1))/(RARF      62
     1(M,1)-RARF(M+1,1))                                                      63
      IF (FF.LT.0.) GO TO 31                                                  64
30    NN1=1                                                                   65
      GO TO 32                                                                66
31    NN1=0                                                                   67
32    IF (NN1.NE.NN) GO TO 110                                                68
33    CONTINUE                                                                69
      COMP=1                                                                  70
      COMP=COMP+.2                                                            71
      RETURN                                                                  72
80    COMP=2                                                                  73
      COMP=COMP+.2                                                            74
      RETURN                                                                  75
90    COMP=3.                                                                 76
      COMP=COMP+.2                                                            77
      RETURN                                                                  78
100   COMP=4.                                                                 79
      COMP=COMP+.2                                                            80
      RETURN                                                                  81
110   COMP=5.                                                                 82
      COMP=COMP+.2                                                            83
      RETURN                                                                  84
      DEBUG SUBCHK
      END                                                                     85
      FUNCTION PICK(ZP,RP,KODE)                                                1
C                                                                              2
C         DETERMINES 2 CLOSEST CONSECUTIVE POINTS ON SPECIFIED                 3
C         LINE OF DISCONTINUITY TO A GIVEN POINT                               4
C                                                                              5
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPS       6
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LEN       7
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS              8
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2       9
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(      10
     115,4),RPART(15,2)                                                       11
```

```
C                                                                       12
C                                                                       13
      COMMON Z0,R0,P0,U0,V0,L0,M0,RHO0,E0,A0,UBAR0,VBAR0                14
C                                                                       15
      COMMON NP,NT,NR,NI,NDEL,ISUB                                      16
C                                                                       17
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                   18
      COMMON DIRCOS                                                     19
      COMMON TIME                                                       20
      COMMON IRARF                                                      21
      COMMON KSTOP                                                      22
      COMMON TPSI                                                       23
      COMMON KKK                                                        24
      REAL L0,M0,LENGTH,MU,K0                                           25
C                                                                       26
      GO TO (5,10,100,300),KODE                                         27
5     NN=NP                                                             28
      K=1                                                               29
      GO TO 15                                                          30
10    NN=NT                                                             31
      K=2                                                               32
15    AA=(TAB(1,1,K)-ZP)**2+(TAB(1,2,K)-RP)**2                          33
C                                                                       34
C          SEARCH SHOCK TABLES                                          35
C                                                                       36
      DO 60 N=2,NN                                                      37
      A=(TAB(N,1,K)-ZP)**2+(TAB(N,2,K)-RP)**2                           38
      IF (A.GE.AA) GO TO 23                                             39
      AA=A                                                              40
60    CONTINUE                                                          41
      PICK=NN-1                                                         42
      PICK=PICK+.2                                                      43
      RETURN                                                            44
23    PICK=N-1                                                          45
      PICK=PICK+.2                                                      46
      RETURN                                                            47
```

```
100   AA=(RARF(1,1)-ZP)**2+(RARF(1,2)-RP)**2                          48
C                                                                     49
C         SEARCH RAREFACTION TABLE                                    50
C                                                                     51
      DO 200 N=2,NR                                                   52
      A=(RARF(N,1)-ZP)**2+(RARF(N,2)-RP)**2                           53
      IF (A.GE.AA) GO TO 203                                          54
      AA=A                                                            55
200   CONTINUE                                                        56
      PICK=NR-1                                                       57
      PICK=PICK+.2                                                    58
      RETURN                                                          59
203   PICK=N-1                                                        60
      PICK=PICK+.2                                                    61
      RETURN                                                          62
300   AA=(SURF(1,1)-ZP)**2+(SURF(1,2)-RP)**2                          63
C                                                                     64
C     SEARCH FREE SURFACE TABLE                                       65
C                                                                     66
      DO 400 N=2,NP                                                   67
      A=(SURF(N,1)-ZP)**2+(SURF(N,2)-RP)**2                           68
      IF (A.GE.AA) GO TO 303                                          69
      AA=A                                                            70
400   CONTINUE                                                        71
      PICK=NP-1                                                       72
      PICK=PICK+.2                                                    73
      RETURN                                                          74
303   PICK=N-1                                                        75
      PICK=PICK+.2                                                    76
304   CONTINUE                                                        77
      RETURN                                                          78
      DEBUG SUBCHK
      END                                                             79
      SUBROUTINE GUESS(KOD1,KOD2,ZP,RP,I2,K2,ZG,RG,DZ,DR)              1
C                                                                     2
C         DETERMINES STARTING POINT AND DELTAS                        3
```

```
C          FOR NEWTON-RAPHSON ITERATION                                          4
C                                                                                5
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPS         6
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LEN         7
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS                8
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2          9
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(        10
     115,4),RPART(15,2)                                                         11
                                                                                12
C                                                                               13
C                                                                               14
      COMMON ZO,RO,PO,UO,VO,LO,MO,RHOO,EO,AO,UBARO,VBARO                        15
C                                                                               16
      COMMON NP,NT,NR,NI,NDEL,ISUB                                              17
C                                                                               18
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                           19
      COMMON DIRCOS                                                             20
      COMMON TIME                                                               21
      COMMON IRARF                                                              22
      COMMON KSTOP                                                              23
      COMMON TPSI                                                               24
      COMMON KKK                                                                25
      REAL LO,MO,LENGTH,MU,KO                                                   26
C                                                                               27
      KS=0                                                                      28
      IF (KOD1.EQ.2) GO TO 10                                                   29
      ZG=TAB(I2,1,K2)                                                           30
      RG=TAB(I2,2,K2)                                                           31
      IF (IRARF.EQ.1) GO TO 9                                                   32
    2 M=1-(NR-2)*(K2-2)                                                         33
      FF=RG-RARF(M+1,2)-(RARF(M,2)-RARF(M+1,2))*(ZG-RARF(M+1,1))/(RARF(M        34
     1,1)-RARF(M+1,1))                                                          35
      IF (FF.GT.0.) GO TO 9                                                     36
      ZG=(1.-.02/(RADIUS-RG))*ZG                                                37
      RG=RG+.02                                                                 38
      IF (RG.GT.(RADIUS-.01)) GO TO 110                                         39
      GO TO 2
```

```
9       CONTINUE                                                        40
        GO TO 50                                                        41
10      JJJ=NT-1                                                        42
        IF (ITS3.EQ.1) GO TO 26                                         43
        DO 24 M=1,JJJ                                                   44
        IF (RP.GT.TAB(M+1,2,2).OR.RP.LT.TAB(M,2,2)) GO TO 24            45
        GO TO 25                                                        46
24      CONTINUE                                                        47
25      CONTINUE                                                        48
        FF=ZP-TAB(M+1,1,2)-(TAB(M,1,2)-TAB(M+1,1,2))*(RP-TAB(M+1,2,2))/(TA  49
       1B(M,2,2)-TAB(M+1,2,2))                                          50
        IF (FF.GT.0.) GO TO 20                                          51
26      CONTINUE                                                        52
        ZG=ZP                                                           53
        RG=RP                                                           54
        GO TO 50                                                        55
20      IK=M                                                            56
        M=CROSS(TAB(IK,1,2),TAB(IK,2,2),TAB(IK+1,1,2),TAB(IK+1,2,2),0.,0.,  57
       1ZP,RP,ZG,RG)                                                    58
        GO TO (50,920,930),M                                            59
C                                                                       60
C           COMPUTE DELTAS                                              61
C                                                                       62
50      DEL=DELTA                                                       63
C                                                                       64
C           Z DELTA                                                     65
C                                                                       66
        LL=0                                                            67
52      DO 70 N=1,NDEL                                                  68
        ZZ=ZG+DEL                                                       69
        M=COMP(ZG,RG,ZZ,RG)                                             70
        IF (M.NE.1) GO TO 55                                            71
        DZ=DEL                                                          72
        GO TO 80                                                        73
55      ZZ=ZG-DEL                                                       74
        M=COMP(ZG,RG,ZZ,RG)                                             75
```

```
      IF (M.NE.1) GO TO 60
      DZ=-DEL
      GO TO 80
60    DEL=.5*DEL
70    CONTINUE
      LL=LL+1
      IF (LL.EQ.3) GO TO 75
      RG=RG-DELTA/5.
      DEL=DELTA
      GO TO 52
75    KOD2=2
      RETURN
80    DEL=DELTA
      IF (KS.EQ.1) GO TO 120
C
C          R DELTA
C
      LL=0
82    DO 100 N=1,NDEL
      RR=RG+DEL
      M=COMP(ZG,RG,ZG,RR)
      IF (M.NE.1) GO TO 85
      DR=DEL
      IF (KS.EQ.1) GO TO 50
      GO TO 120
85    RR=RG-DEL
      M=COMP(ZG,RG,ZG,RR)
      IF (M.NE.1) GO TO 90
      DR=-DEL
      IF (KS.EQ.1) GO TO 50
      GO TO 120
90    DEL=.5*DEL
100   CONTINUE
108   CONTINUE
      LL=LL+1
      FLL=LL
```

```
      IF (LL.EQ.5) GO TO 110                                                112
C                                                                           113
104   ZG1=ZG+DELTA/5.*FLL*(-1.)**LL                                         114
      M=COMP(ZG,RG,ZG1,RG)                                                  115
      ZG=ZG1                                                                116
      IF (M.NE.1) GO TO 108                                                 117
      KS=1                                                                  118
      GO TO 82                                                              119
110   KOD2=2                                                                120
      RETURN                                                                121
120   KOD2=1                                                                122
      RETURN                                                                123
920   WRITE (3,922)                                                         124
922   FORMAT (42HOERROR FOR COINCIDENT LINES IN SUBR. GUESS)                125
      GO TO 950                                                             126
C                                                                           127
930   WRITE (3,932)                                                         128
932   FORMAT (40HOERROR FOR PARALLEL LINES IN SUBR. GUESS)                  129
C                                                                           130
950   WRITE (3,952) KOD1,I2,K2,ZP,RP                                        131
952   FORMAT (1X5HKOD1=,I4,4X3HI2=,I4,4X3HK2=,I4/1X3HZP=,E15.8,4X3HRP=,E    132
     115.8/1H1)                                                             133
      XYZ=-2.                                                               134
      ZYX=SQRT(XYZ)                                                         135
      CALL EXIT                                                             136
      RETURN                                                                137
      END                                                                   138
      FUNCTION CROSS(X1,Y1,X2,Y2,X3,Y3,X4,Y4,X,Y)                             1
C                                                                             2
C         FINDS INTERSECTION OF TWO STRAIGHT LINES                            3
C                                                                             4
      EPS=.0000001                                                            5
      A1=Y2-Y1                                                                6
      B1=X1-X2                                                                7
      C1=X1*A1+Y1*B1                                                          8
      A2=Y4-Y3                                                                9
```

```
      B2=X3-X4                                                          10
      C2=X3*A2+Y3*B2                                                    11
      DET=A1*B2-A2*B1                                                   12
      D1=C1*B2-C2*B1                                                    13
      D2=A1*C2-A2*C1                                                    14
      IF (ABS(DET).LE.EPS) GO TO 10                                     15
      X=D1/DET                                                          16
      Y=D2/DET                                                          17
      CROSS=1                                                           18
      CROSS=CROSS+.2                                                    19
      RETURN                                                            20
10    IF (ABS(D1).GT.EPS) GO TO 20                                      21
      IF (ABS(D2).LE.EPS) GO TO 30                                      22
20    CROSS=3                                                           23
      CROSS=CROSS+.2                                                    24
      RETURN                                                            25
30    X=X1                                                              26
      Y=Y1                                                              27
      CROSS=2                                                           28
      CROSS=CROSS+.2                                                    29
316   CONTINUE                                                          30
      RETURN                                                            31
      DEBUG SUBCHK
      END                                                               32
      FUNCTION PART(MODE,ZP,RP,ZX,RX,DELTA,NDEL)                         1
C                                                                        2
C         LOCATES A POINT IN THE SAME REGION AS A GIVEN POINT            3
C         TO BE USED IN COMPUTING A PARTIAL                              4
C     MODE=1,WITH RESPECT TO R                                           5
C     MODE=2,WITH RESPECT TO Z                                           6
C                                                                        7
      GO TO (2,4),MODE                                                   8
2     DR=DELTA                                                           9
      DZ=0.                                                             10
      GO TO 8                                                           11
4     DR=0.                                                             12
```

```
      DZ=DELTA                                                          13
8     DO 50 NN=1,NDEL                                                   14
      RR=RP+DR                                                          15
      ZZ=ZP+DZ                                                          16
      M=COMP(ZP,RP,ZZ,RR)                                               17
      IF (M.EQ.1) GO TO 60                                              18
      RR=RP-DR                                                          19
      ZZ=ZP-DZ                                                          20
      M=COMP(ZP,RP,ZZ,RR)                                               21
      IF (M.EQ.1) GO TO 60                                              22
      DZ=DZ*.5                                                          23
      DR=DR*.5                                                          24
50    CONTINUE                                                          25
      PART=2                                                            26
      PART=PART+.2                                                      27
      RETURN                                                            28
60    ZX=ZZ                                                             29
      RX=RR                                                             30
      PART=1                                                            31
      PART=PART+.2                                                      32
      RETURN                                                            33
      DEBUG SUBCHK
      END                                                               34
      FUNCTION PICK2(ZP,RP,KODE)                                         1
C                                                                        2
C         DETERMINES 2 CLOSEST CONSECUTIVE POINTS ON SPECIFIED           3
C         LINE OF DISCONTINUITY TO A GIVEN POINT                         4
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPS  5
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LEN  6
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS         7
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2  8
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(  9
     115,4),RPART(15,2)                                                  10
C                                                                        11
      COMMON ZO,RO,PO,UO,VO,LO,MO,RHOO,EO,AO,UBARO,VBARO                 12
      COMMON NP,NT,NR,NI,NDEL,ISUB                                       13
```

///

```
C                                                                       14
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                   15
      COMMON DIRCOS                                                     16
      COMMON TIME                                                       17
      COMMON IRARF                                                      18
      COMMON KSTOP                                                      19
      COMMON TPSI                                                       20
      COMMON KKK                                                        21
      REAL L0,M0,LENGTH,MU,K0                                           22
C                                                                       23
      GO TO (5,10,100,205,210,300,500),KODE                             24
5     NN=NP                                                             25
      K=1                                                               26
      GO TO 15                                                          27
10    NN=NT                                                             28
      K=2                                                               29
15    AA=(TAB2(1,1,K)-ZP)**2+(TAB2(1,2,K)-RP)**2                        30
C                                                                       31
C               SEARCH SHOCK TABLES                                     32
C                                                                       33
      DO 60 N=2,NN                                                      34
      A=(TAB2(N,1,K)-ZP)**2+(TAB2(N,2,K)-RP)**2                         35
      IF (A.GE.AA) GO TO 23                                             36
      AA=A                                                              37
60    CONTINUE                                                          38
      PICK2=NN-1                                                        39
      PICK2=PICK2+.2                                                    40
      RETURN                                                            41
23    PICK2=N-1                                                         42
      PICK2=PICK2+.2                                                    43
      RETURN                                                            44
100   AA=(RARF2(1,1)-ZP)**2+(RARF2(1,2)-RP)**2                          45
C                                                                       46
C               SEARCH RAREFACTION TABLE                                47
C                                                                       48
      DO 200 N=2,NR                                                     49
```

```
      A=(RARF2(N,1)-ZP)**2+(RARF2(N,2)-RP)**2                          50
      IF (A.GE.AA) GO TO 203                                           51
      AA=A                                                             52
200   CONTINUE                                                         53
      PICK2=NR-1                                                       54
      PICK2=PICK2+.2                                                   55
      RETURN                                                           56
203   PICK2=N-1                                                        57
      PICK2=PICK2+.2                                                   58
      RETURN                                                           59
205   NN=NP                                                            60
      K=1                                                              61
      GO TO 215                                                        62
210   NN=NT                                                            63
      K=2                                                              64
215   AA=(SPART(1,1,K)-ZP)**2+(SPART(1,2,K)-RP)**2                     65
C                                                                      66
C               SEARCH SHOCK PARTICLE TABLES                           67
C                                                                      68
      DO 260 N=2,NN                                                    69
      A=(SPART(N,1,K)-ZP)**2+(SPART(N,2,K)-RP)**2                      70
      IF (A.GE.AA) GO TO 223                                           71
      AA=A                                                             72
260   CONTINUE                                                         73
      PICK2=NN-1                                                       74
      PICK2=PICK2+.2                                                   75
      RETURN                                                           76
223   PICK2=N-1                                                        77
      PICK2=PICK2+.2                                                   78
      RETURN                                                           79
300   AA=(RPART(1,1)-ZP)**2+(RPART(1,2)-RP)**2                         80
C                                                                      81
C               SEARCH RAREFACTION PARTICLE TABLE                      82
C                                                                      83
      DO 400 N=2,NR                                                    84
      A=(RPART(N,1)-ZP)**2+(RPART(N,2)-RP)**2                          85
```

```
      IF (A.GE.AA) GO TO 403                                                 86
      AA=A                                                                   87
400   CONTINUE                                                               88
      PICK2=NR-1                                                             89
      PICK2=PICK2+.2                                                         90
      RETURN                                                                 91
403   PICK2=N-1                                                              92
      PICK2=PICK2+.2                                                         93
      RETURN                                                                 94
500   AA=(SURF2(1,1)-ZP)**2+(SURF2(1,2)-RP)**2                               95
C                                                                            96
C     SEARCH FREE SURFACE TABLE                                              97
C                                                                            98
      DO 520 N=2,NP                                                          99
      A=(SURF2(N,1)-ZP)**2+(SURF2(N,2)-RP)**2                               100
      IF (A.GE.AA) GO TO 523                                                101
      AA=A                                                                  102
520   CONTINUE                                                              103
      PICK2=NP-1                                                            104
      PICK2=PICK2+.2                                                        105
      RETURN                                                                106
523   PICK2=N-1                                                             107
      PICK2=PICK2+.2                                                        108
      RETURN                                                                109
      DEBUG SUBCHK
      END                                                                   110
      FUNCTION TEST(ZP,RP)                                                    1
C                                                                             2
C                                                                             3
C         DETERMINES IF A GIVEN INTERIOR POINT IS IN                          4
C         THE REGION TO BE CONSIDERED                                         5
C                                                                             6
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPS      7
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LEN      8
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS             9
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2      10
```

```
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(
     115,4),RPART(15,2)
C
C
      COMMON ZO,RO,PO,UO,VO,LO,MO,RHOO,EO,AO,UBARO,VBARO
C
      COMMON NP,NT,NR,NI,NDEL,ISUB
C
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H
      COMMON DIRCOS
      COMMON TIME
      COMMON IRARF
      COMMON KSTOP
      COMMON TPSI
      COMMON KKK
      REAL LO,MO,LENGTH,MU,KO
C
      EPS=.0001
      IF (ITS3.EQ.1) GO TO 5
      IF (RP.GT.(RADIUS+EPS)) GO TO 5
      M=PICK2(ZP,RP,7)
      FF=ZP-SURF2(M+1,1)-(SURF2(M,1)-SURF2(M+1,1))*(RP-SURF2(M,2))/(SURF
     12(M,2)-SURF2(M+1,2))
      KICK=5
      CALL DVCHK(KQ)
      IF (KQ.EQ.1) GO TO 9980
      IF (FF)  200,5,5
5     CONTINUE
      IF (ZP.GT.-EPS) GO TO 1
      IF (RP.GT.(RADIUS+EPS)) GO TO 200
1     DO 10 K=1,2
      IF (ITS3.EQ.1) GO TO 100
      M=PICK2(ZP,RP,K)
      IF (TAB2(M,2,K).GT.RP.AND.M.NE.1)M=M-1
      FF=ZP-TAB2(M+1,1,K)-(TAB2(M,1,K)-TAB2(M+1,1,K))*(RP-TAB2(M+1,2,K))
     1/(TAB2(M,2,K)-TAB2(M+1,2,K))
```

```
      KICK=50                                                          47
      CALL DVCHK(KQ)                                                   48
      IF (KQ.EQ.1) GO TO 9980                                          49
      IF (K.EQ.2) GO TO 50                                             50
      IF (RP.GT.RADIUS) GO TO 10                                       51
      IF (FF)  200,10,10                                               52
50    IF (FF)  10,10,200                                               53
10    CONTINUE                                                         54
      DO 20 K=1,2                                                      55
      J=K+3                                                            56
      M=PICK2(ZP,RP,J)                                                 57
      IF (SPART(M,2,K).GT.RP.AND.M.NE.1)M=M-1                          58
      FF=ZP-SPART(M+1,1,K)-(SPART(M,1,K)-SPART(M+1,1,K))*(RP-SPART(M+1,2   59
     1,K))/(SPART(M,2,K)-SPART(M+1,2,K))                               60
      KICK=15                                                          61
      CALL DVCHK(KQ)                                                   62
      IF (KQ.EQ.1) GO TO 9980                                          63
      IF (K.EQ.2) GO TO 15                                             64
      IF (RP.GT.RADIUS) GO TO 20                                       65
      IF (FF.LT..001) GO TO 300                                        66
      GO TO 20                                                         67
15    IF (FF.GT.-.001) GO TO 400                                       68
20    CONTINUE                                                         69
      IF (IRARF.EQ.1) GO TO 100                                        70
      M=PICK2(ZP,RP,3)                                                 71
      FF=RP-RARF2(M+1,2)-(RARF2(M,2)-RARF2(M+1,2))*(ZP-RARF2(M+1,1))/(RA   72
     1RF2(M,1)-RARF2(M+1,1))                                           73
      KICK=20                                                          74
      CALL DVCHK(KQ)                                                   75
      IF (KQ.EQ.1) GO TO 9980                                          76
      IF (FF.LT.0.) GO TO 100                                          77
      M=PICK2(ZP,RP,6)                                                 78
      FF=RP-RPART(M+1,2)-(RPART(M,2)-RPART(M+1,2))*(ZP-RPART(M+1,1))/(RP   79
     1ART(M,1)-RPART(M+1,1))                                           80
      KICK=100                                                         81
      CALL DVCHK(KQ)                                                   82
```

```
      IF (KQ.EQ.1) GO TO 9980                                              83
      IF (FF.LT.0.) GO TO 500                                              84
100   TEST=1                                                               85
      TEST=TEST+.2                                                         86
      RETURN                                                               87
200   TEST=2                                                               88
      TEST=TEST+.2                                                         89
      RETURN                                                               90
300   TEST=3                                                               91
      TEST=TEST+.2                                                         92
      RETURN                                                               93
400   TEST=4                                                               94
      TEST=TEST+.2                                                         95
      RETURN                                                               96
500   TEST=5                                                               97
      TEST=TEST+.2                                                         98
      RETURN                                                               99
9980  WRITE (3,9985) KICK                                                 100
9985  FORMAT (32HODIVIDE CHECK NEAR STATEMENT NO.,I5,14H IN SUBR. TEST/1  101
     1H1)                                                                 102
      RETURN                                                              103
      DEBUG SUBCHK
      END                                                                 104
      SUBROUTINE FGOF5(Z5,R5,SS,QQ,PHI,NMAX,MMAX)
C     SUBROUTINE FGOF5(Z5,R5,SS,QQ)
C                                                                           2
C         COMPUTES S5,Q5 FOR INTERIOR REGION                                3
C         ITERATION FOR Z5,R5                                               4
C                                                                           5
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPS    6
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LEN    7
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS           8
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2     9
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(    10
     115,4),RPART(15,2)                                                     11
C                                                                           12
```

```
C                                                                       13
      COMMON Z0,R0,P0,U0,V0,L0,M0,RHO0,E0,A0,UBAR0,VBAR0                14
C                                                                       15
      COMMON NP,NT,NR,NI,NDEL,ISUB                                      16
C                                                                       17
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                   18
      COMMON DIRCOS                                                     19
      COMMON TIME                                                       20
      COMMON IRARF                                                      21
      COMMON KSTOP                                                      22
      COMMON TPSI                                                       23
      COMMON KKK                                                        24
      REAL L0,M0,LENGTH,MU,K0                                           25
      DOUBLE PRECISION PHI(20,20,6)
C                                                                       26
      DIMENSION ANS(6)                                                  27
C                                                                       28
      TSSS=1.1
      IF(TIME.GT.TSSS) GO TO 100
      CALL DBLTRP(Z5,R5,ANS)
      GO TO 101
  100 CALL DSURFT(Z5,R5,ANS,PHI,NMAX,MMAX)
  101 U5=ANS(2)
C     U5=ANS(2)
      V5=ANS(3)                                                         31
      SS=Z5-Z0+H*V5                                                     32
      QQ=R5-R0+H*U5                                                     33
      RETURN                                                            34
      DEBUG SUBCHK
      END                                                               35
      SUBROUTINE ITRP                                                    1
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPS 2
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LEN 3
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS        4
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2 5
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2( 6
```

```
     115,4),RPART(15,2)                                                    7
C                                                                          8
C                                                                          9
      COMMON ZO,RO,PO,UO,VO,LO,MO,RHOO,EO,AO,UBARO,VBARO                  10
C                                                                         11
      COMMON NP,NT,NR,NI,NDEL,ISUB                                        12
C                                                                         13
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                     14
      COMMON DIRCOS                                                       15
      COMMON TIME                                                         16
      COMMON IRARF                                                        17
      COMMON KSTOP                                                        18
      COMMON TPSI                                                         19
      COMMON KKK                                                          20
C                                                                         21
C                                                                         22
      REAL LO,MO,LENGTH,MU,KO                                             23
C                                                                         24
C INTERPOLATION SCHEME FOR POINTS BETWEEN PARTICLE CURVES                 25
C AND DISCONTINUITIES                                                     26
C                                                                         27
      EPS=.0000001                                                        28
      KOD1=0                                                              29
      KOD2=0                                                              30
      DO 1000 J=1,20                                                      31
      DO 1000 I=1,20                                                      32
      M=TEST(Z(I),R(J))                                                   33
      IF (M.NE.5) GO TO 906                                               34
      IF (IRARF.EQ.1) GO TO 906                                           35
      NR1=NR-1                                                            36
      DO 907 JJ=1,NR1                                                     37
      IF (RARF2(JJ,1).LT.Z(I).AND.RARF2(JJ+1,1).GT.Z(I)) GO TO 908        38
907   CONTINUE                                                            39
908   CONTINUE                                                            40
      FF=R(J)-RARF2(JJ+1,2)+(RARF2(JJ,2)-RARF2(JJ+1,2))*(Z(I)-RARF2(JJ+1  41
     1,1))/(RARF2(JJ,1)-RARF2(JJ+1,1))                                    42
```

```
      IF (FF.LT.0.) GO TO 1000                                          43
906   CONTINUE                                                          44
      IF (M.EQ.3.AND.Z(I).LT.EPS.AND.ABS(R(J)-RADIUS).LT.EPS) GO TO 1000 45
      IF (M.LT.3) GO TO 8051                                            46
      WRITE (3,8050)                                                    47
8050  FORMAT (30HOINTERPOLATION SCHEME EMPLOYED)                        48
614   FORMAT (1X5HZO  =,E15.8,4X5HRO  =,E15.8)                          49
      ZO=Z(I)                                                           50
      RO=R(J)                                                           51
      WRITE (3,614) ZO,RO                                               52
      WRITE (3,8888) M                                                  53
8888  FORMAT (1X4HM  =,I4)                                              54
8051  CONTINUE                                                          55
      GO TO (1000,1000,1001,1010,1100),M                                56
1001  IF (IRARF.EQ.1) GO TO 1003                                        57
      M=PICK2(Z(I),R(J),3)                                              58
      FF=R(J)-RARF2(M+1,2)+(RARF2(M,2)-RARF2(M+1,2))*(Z(I)-RARF2(M+1,1)) 59
     1/(RARF2(M,1)-RARF2(M+1,1))                                        60
      IF (FF.GT.0.) GO TO 1003                                          61
      DO 8002 K=1,6                                                     62
      L=K+2                                                             63
8002  XMESH2(I,J,K)=RARF(1,L)                                           64
      GO TO 1000                                                        65
1003  L=I+1                                                             66
      N=TEST(Z(L),R(J))                                                 67
      GO TO (1004,1007,1030,1030,1050),N                                68
1004  M=PICK2(Z(I),R(J),1)                                              69
      DO 1006 K=1,8                                                     70
      ANS=TAB2(M,K,1)+(TAB2(M+1,K,1)-TAB2(M,K,1))*(R(J)-TAB2(M,2,1))/(TA 71
     1B2(M+1,2,1)-TAB2(M,2,1))                                          72
      IF (K.NE.1) GO TO 8005                                            73
      ANS1=ANS                                                          74
      GO TO 1006                                                        75
8005  CONTINUE                                                          76
      IF (K.EQ.2) GO TO 1006                                            77
      KX=K-2                                                            78
```

```
      IF (ABS(R(J)-RADIUS).GT.EPS) GO TO 9024                                79
      IF (K.GT.5.OR.K.LT.4) GO TO 9024                                       80
      ANS=TAB2(M+1,K,1)                                                      81
9024  CONTINUE                                                               82
      XMESH2(I,J,KX)=ANS+(XMESH2(L,J,KX)-ANS)*(Z(I)-ANS1)/(Z(L)-ANS1)        83
      IF (ABS(R(J)-RADIUS).GT.EPS) GO TO 1006                                84
      XMESH2(I,J,1)=0.                                                       85
      XMESH2(I,J,4)=RHOSTR                                                   86
      XMESH2(I,J,5)=0.                                                       87
      XMESH2(I,J,6)=SQRT(BIGAPR/RHOSTR)                                      88
1006  CONTINUE                                                               89
      GO TO 1000                                                             90
1007  MM=PICK2(Z(I),R(J),2)                                                  91
      M=PICK2(Z(I),R(J),1)                                                   92
      DO 1009 K=1,8                                                          93
      ANS=TAB2(M,K,1)+(TAB2(M+1,K,1)-TAB2(M,K,1))*(R(J)-TAB2(M,2,1))/(TA     94
     1B2(M+1,2,1)-TAB2(M,2,1))                                               95
      ANSW=TAB2(MM,K,2)+(TAB2(MM+1,K,2)-TAB2(MM,K,2))*(R(J)-TAB2(MM,2,2)     96
     1)/(TAB2(MM+1,2,2)-TAB2(MM,2,1))                                        97
      IF (K.NE.1) GO TO 1008                                                 98
      ANS1=ANS                                                               99
      ANS2=ANSW                                                             100
      GO TO 1009                                                            101
1008  CONTINUE                                                              102
      IF (K.EQ.2) GO TO 1009                                                103
      KX=K-2                                                                104
      XMESH2(I,J,KX)=ANS+(ANSW-ANS)*(Z(I)-ANS1)/(ANS2-ANS1)                 105
1009  CONTINUE                                                              106
      GO TO 1000                                                            107
1010  IF (IRARF.EQ.1) GO TO 1013                                            108
      M=PICK2(Z(I),R(J),3)                                                  109
      FF=R(J)-RARF2(M+1,2)+(RARF2(M,2)-RARF2(M+1,2))*(Z(I)-RARF2(M+1,1))    110
     1/(RARF2(M,1)-RARF2(M+1,1))                                            111
      IF (FF.GT.0.) GO TO 1013                                              112
      DO 1012 K=1,6                                                         113
      L=K+2                                                                 114
```

```
1012  XMESH2(I,J,K)=RARF(1,L)                                                115
      GO TO 1000                                                             116
1013  L=I-1                                                                  117
      IF (ABS(Z(I)).LT.EPS) GO TO 1017                                       118
      N=TEST(Z(L),R(J))                                                      119
      GO TO (1014,1017,1030,1017,1051),N                                     120
1014  M=PICK2(Z(I),R(J),2)                                                   121
      DO 1016 K=1,8                                                          122
      ANS=TAB2(M,K,2)+(TAB2(M+1,K,2)-TAB2(M,K,2))*(R(J)-TAB2(M,2,2))/(TA     123
     1B2(M+1,2,2)-TAB2(M,2,2))                                               124
      IF (K.NE.1) GO TO 1015                                                 125
      ANS1=ANS                                                               126
      GO TO 1016                                                             127
1015  CONTINUE                                                               128
      IF (K.EQ.2) GO TO 1016                                                 129
      KX=K-2                                                                 130
      XMESH2(I,J,KX)=ANS+(XMESH2(L,J,KX)-ANS)*(Z(I)-ANS1)/(Z(L)-ANS1)        131
1016  CONTINUE                                                               132
      GO TO 1000                                                             133
1017  L=J-1                                                                  134
      N=TEST(Z(I),R(L))                                                      135
      GO TO (1024,1030,1030,1030,1051),N                                     136
1024  M=PICK2(Z(I),R(J),2)                                                   137
      DO 1026 K=2,8                                                          138
      ANS=TAB2(M,K,2)+(TAB2(M+1,K,2)-TAB2(M,K,2))*(Z(I)-TAB2(M,1,2))/(TA     139
     1B2(M+1,1,2)-TAB2(M,1,2))                                               140
      IF (K.NE.2) GO TO 1025                                                 141
      ANS1=ANS                                                               142
      GO TO 1026                                                             143
1025  CONTINUE                                                               144
      KX=K-2                                                                 145
      IF (ABS(Z(I)).GT.EPS) GO TO 9125                                       146
      IF (K.GT.5.OR.K.LT.4) GO TO 9125                                       147
      ANS=TAB2(M+1,K,2)                                                      148
9125  CONTINUE                                                               149
      XMESH2(I,J,KX)=ANS+(XMESH2(I,L,KX)-ANS)*(R(J)-ANS1)/(R(L)-ANS1)        150
```

```
      IF (ABS(Z(I)).GT.EPS) GO TO 1026                                  151
      XMESH2(I,J,1)=0.                                                  152
      XMESH2(I,J,4)=RHOSTR                                              153
      XMESH2(I,J,5)=0.                                                  154
      XMESH2(I,J,6)=SQRT(BIGAPR/RHOSTR)                                 155
1026  CONTINUE                                                          156
      GO TO 1000                                                        157
1100  M=PICK2(Z(I),R(J),3)                                              158
      ANS1=RARF2(M+1,2)+(RARF2(M,2)-RARF2(M+1,2))*(Z(I)-RARF2(M+1,1))/(R 159
     1ARF2(M,1)-RARF2(M+1,1))                                           160
      L=J+1                                                             161
      M=TEST(Z(I),R(L))                                                 162
      GO TO (1101,1030,1051,1030),M                                     163
1101  CONTINUE                                                          164
      DO 1106 K=1,6                                                     165
      LL=K+2                                                            166
      XMESH2(I,J,K)=RARF(1,LL)+(XMESH2(I,L,K)-RARF(1,LL))*(R(J)-ANS1)/(R 167
     1(L)-ANS1)                                                         168
1106  CONTINUE                                                          169
      GO TO 1000                                                        170
1050  JJ1=J                                                             171
      II1=I                                                             172
      KOD1=1                                                            173
      GO TO 1000                                                        174
1051  JJ2=J                                                             175
      II2=I                                                             176
      KOD2=1                                                            177
1000  CONTINUE                                                          178
      IF (KOD1.EQ.0) GO TO 2000                                         179
      M=PICK2(Z(II1),R(JJ1),1)                                          180
      DO 2016 K=1,8                                                     181
      ANS=TAB2(M,K,1)+(TAB2(M+1,K,1)-TAB2(M,K,1))*(R(JJ1)-TAB2(M,2,1))/( 182
     1TAB2(M+1,2,1)-TAB2(M,2,1))                                        183
      IF (K.NE.1) GO TO 2005                                            184
      ANS1=ANS                                                          185
      GO TO 2016                                                        186
```

```
2005  CONTINUE                                                                187
      I=III                                                                   188
      J=JJ1                                                                   189
      L=III+1                                                                 190
      IF (K.EQ.2) GO TO 2016                                                  191
      KX=K-2                                                                  192
      XMESH2(I,J,KX)=ANS+(XMESH2(L,J,KX)-ANS)*(Z(I)-ANS1)/(Z(L)-ANS1)         193
2016  CONTINUE                                                                194
2000  IF (KOD2.EQ.0) GO TO 3000                                               195
      M=PICK2(Z(II2),R(JJ2),2)                                                196
      DO 3016 K=1,8                                                           197
      ANS=TAB2(M,K,2)+(TAB2(M+1,K,2)-TAB2(M,K,2))*(R(JJ2)-TAB2(M,2,2))/(      198
     1TAB2(M+1,2,2)-TAB2(M,2,2))                                              199
      IF (K.NE.1) GO TO 3015                                                  200
      ANS1=ANS                                                                201
      GO TO 3016                                                              202
3015  CONTINUE                                                                203
      I=II2                                                                   204
      J=JJ2                                                                   205
      L=II2-1                                                                 206
      IF (K.EQ.2) GO TO 3016                                                  207
      KX=K-2                                                                  208
      XMESH2(I,J,KX)=ANS+(XMESH2(L,J,KX)-ANS)*(Z(I)-ANS1)/(Z(L)-ANS1)         209
3016  CONTINUE                                                                210
3000  CONTINUE                                                                211
      GO TO 3017                                                              212
1030  WRITE (3,1040) I,J                                                      213
1040  FORMAT (27H0 TIME STEP TOO LARGE AT I=,I4,2HJ=,I4,///)                  214
      CALL EXIT                                                               215
3017  CONTINUE                                                                216
C                                                                             217
C                                                                             218
      RETURN                                                                  219
      DEBUG SUBCHK
      END                                                                     220
      SUBROUTINE EXIT                                                           1
```

```
C                                                                       2
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPS  3
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LEN  4
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS         5
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2   6
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(  7
     115,4),RPART(15,2)                                                   8
C                                                                         9
C                                                                        10
      COMMON ZO,RO,PO,UO,VO,LO,MO,RHOO,EO,AO,UBARO,VBARO                 11
C                                                                        12
      COMMON NP,NT,NR,NI,NDEL,ISUB                                       13
C                                                                        14
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                    15
      COMMON DIRCOS                                                      16
      COMMON TIME                                                        17
      COMMON IRARF                                                       18
      COMMON KSTOP                                                       19
      COMMON TPSI                                                        20
      COMMON KKK                                                         21
      REAL LO,MO,LENGTH,MU,KO                                            22
      KSTOP=1                                                            23
      STOP                                                               24
      END                                                                25
      SUBROUTINE FGOFI(ZX,RX,SS,QQ,PHI,NMAX,MMAX)
C     SUBROUTINE FGOFI(ZX,RX,SS,QQ)
C                                                                         2
C         COMPUTES SI,QI FOR INTERIOR REGION                              3
C         ITERATION FOR ZI,RI                                             4
C                                                                         5
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPS  6
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LEN  7
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS         8
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2   9
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(  10
     115,4),RPART(15,2)                                                   11
```

```
C                                                                       12
C                                                                       13
      COMMON Z0,R0,P0,U0,V0,L0,M0,RH00,E0,A0,UBAR0,VBAR0                14
C                                                                       15
      COMMON NP,NT,NR,NI,NDEL,ISUB                                      16
C                                                                       17
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                   18
      COMMON DIRCOS                                                     19
      COMMON TIME                                                       20
      COMMON IRARF                                                      21
      COMMON KSTOP                                                      22
      COMMON TPSI                                                       23
      COMMON KKK                                                        24
      REAL L0,M0,LENGTH,MU,K0                                           25
C                                                                       26
      DIMENSION ANS(6),PSI(4),SPSI(11),CPSI(11)                         27
      DOUBLE PRECISION PHI(20,20,6)
C                                                                       28
C                                                                       29
C                                                                       30
C                                                                       31
C                                                                       32
      TSSS=1.1
      IF(TIME.GT.TSSS) GO TO 100
      CALL DBLTRP(ZX,RX,ANS)
      GO TO 101
  100 CALL DSURFT(ZX,RX,ANS,PHI,NMAX,MMAX)
  101 UI=ANS(2)
C     UI=ANS(2)
      VI=ANS(3)                                                         35
      AI=ANS(6)                                                         36
      SS=ZX-Z0+H*(VI+AI*SIN(TPSI))                                      37
      QQ=RX-R0+H*(UI+AI*COS(TPSI))                                      38
      RETURN                                                            39
```

II.2. Method of Characteristics Calculations and the Computer Code for Materials with Arbitrary Equations of State.

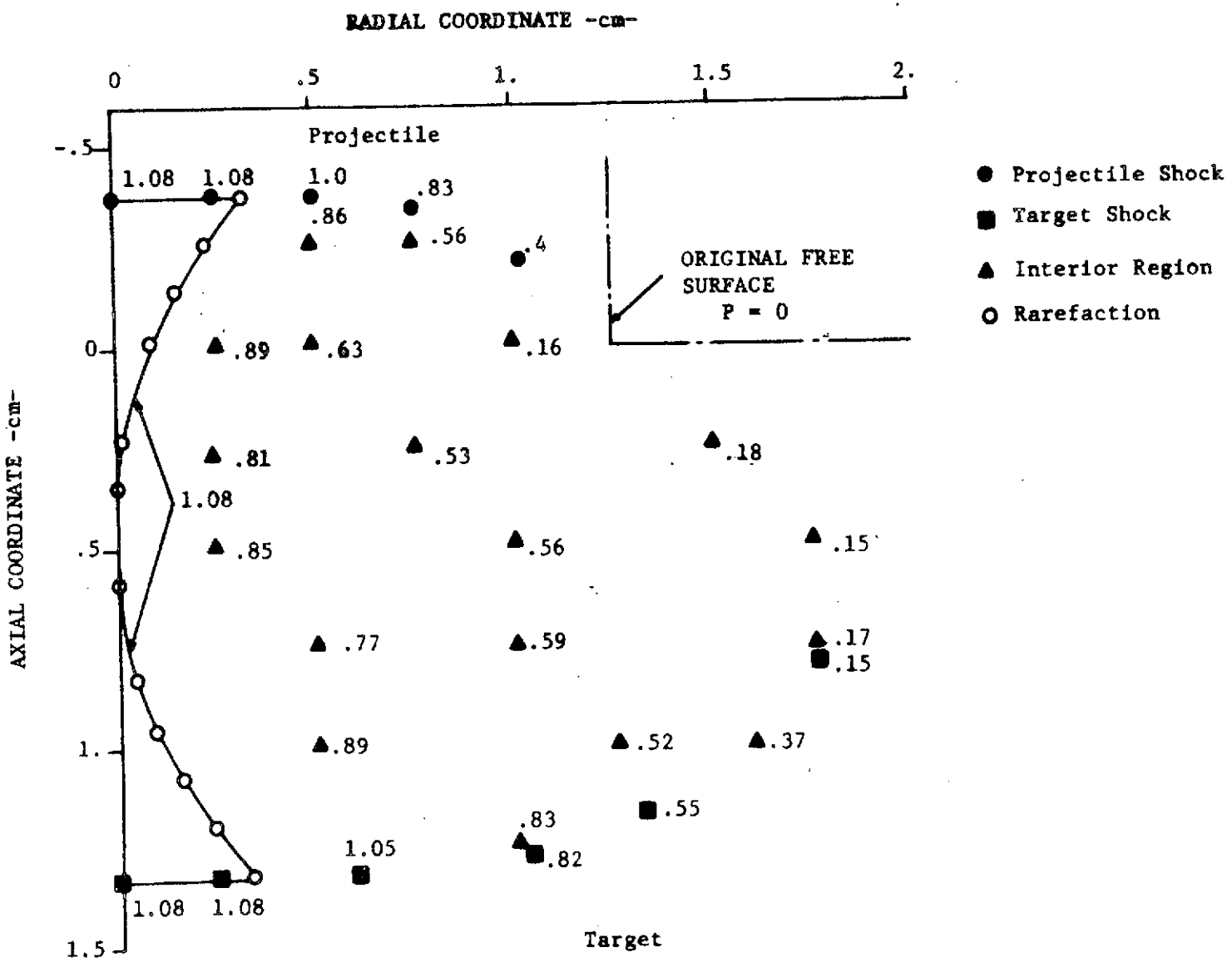

Fig. 1. Pressure distribution from the characteristic solution at t = 1.25 μsec after impact of a 2.5 -cm- diameter projectile at 0.76 cm/μsec on an aluminum half-space.

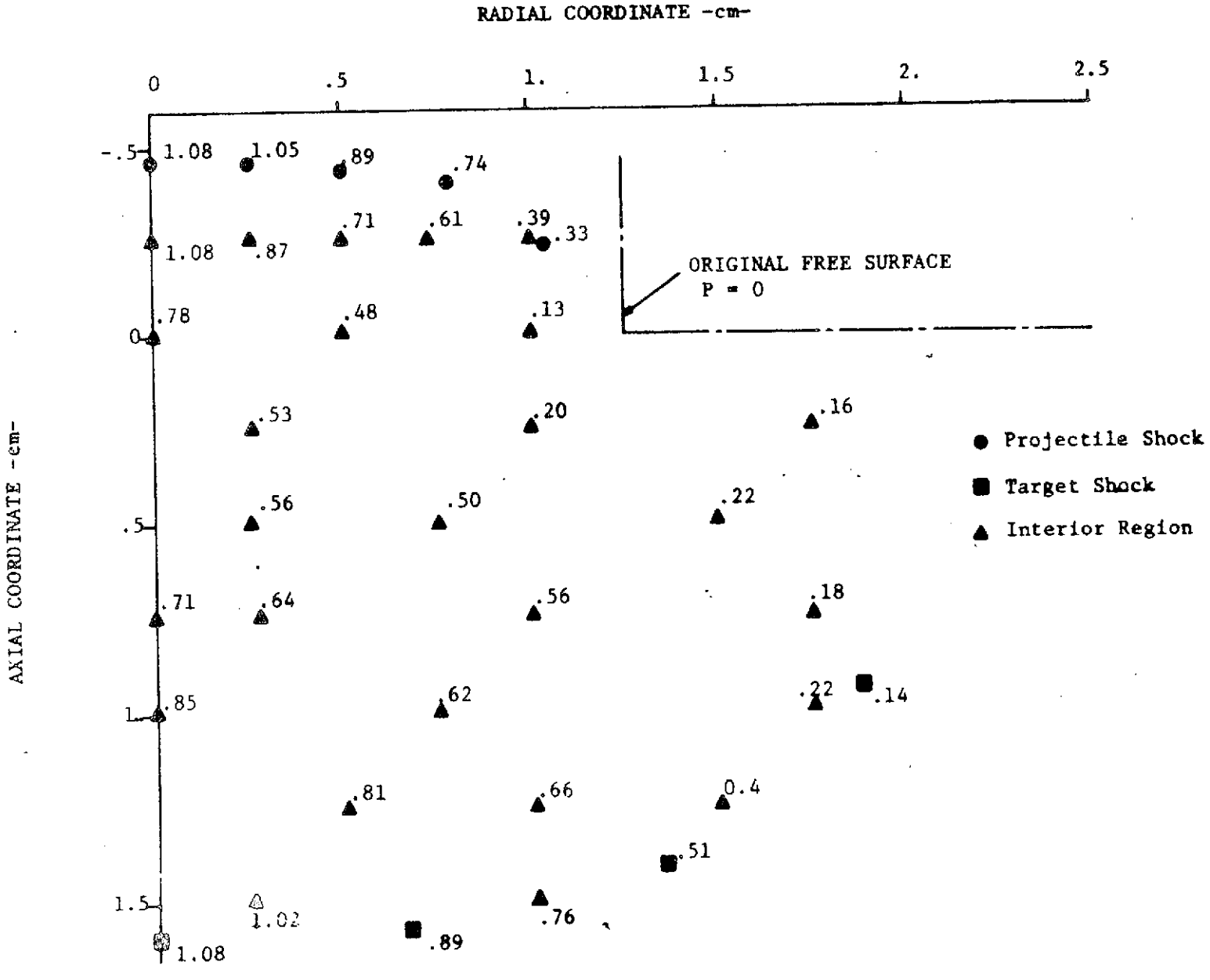

Fig. 2. Pressure distribtuion from the characteristic solution at t = 1.52 μsec after impact of a 2.5 -cm- diameter projectile at 0.76 cm/μsec on an aluminum half-space.

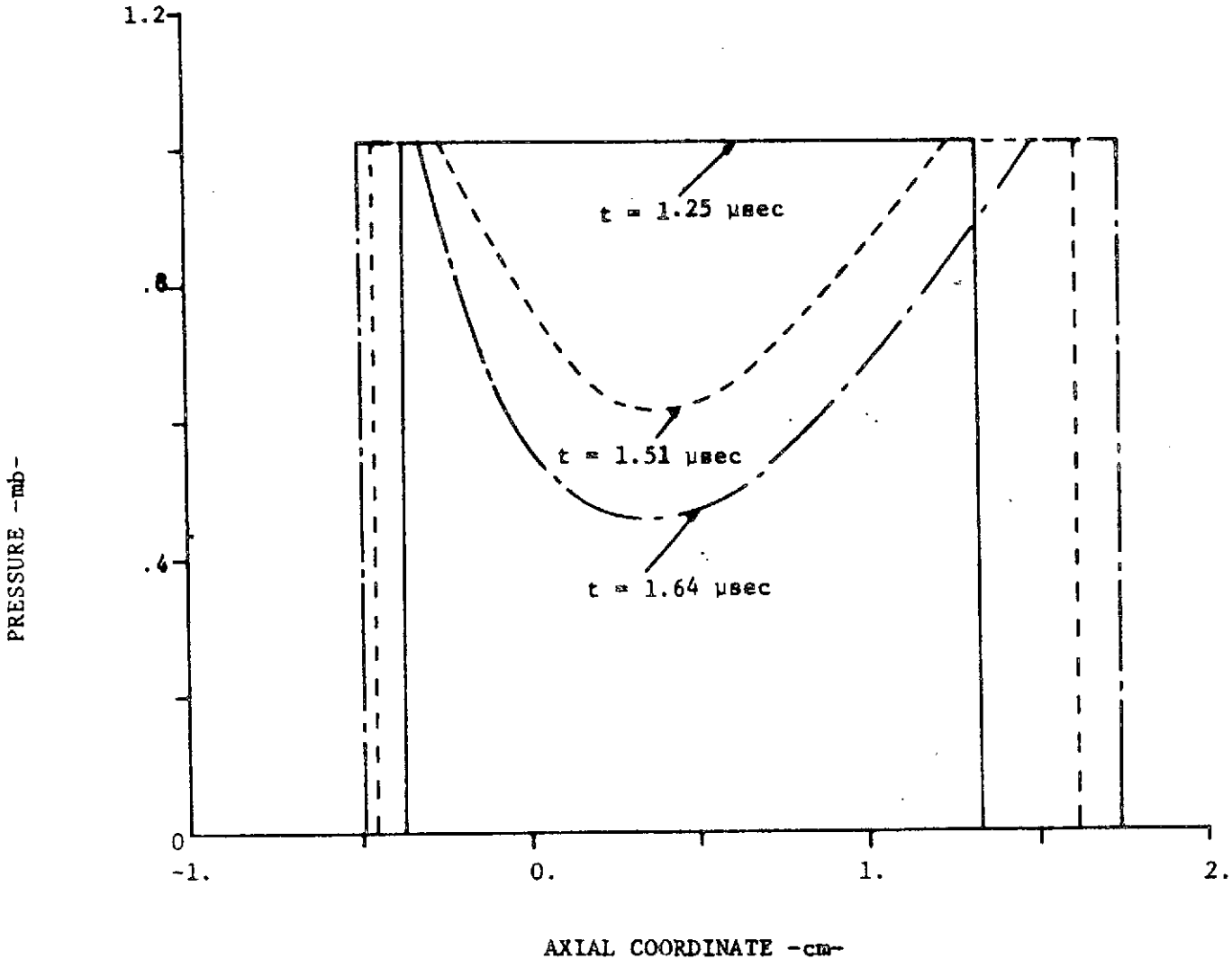

Fig 3. Axial pressure profiles at various times as predicted by the characteraitc method for a 2.5 -cm- diameter aluminum projectile impacting on aluminum half-space at 0.76 cm/μsec.

```
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPSMAIN    1
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LENMAIN    2
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS          MAIN    3
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2MAIN    4
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(MAIN    5
     115,4),RPART(15,2)                                                  MAIN    6
C                                                                        MAIN    7
C                                                                        MAIN    8
      COMMON ZO,RO,PO,UO,VO,LO,MO,RHOO,EO,AO,UBARO,VBARO                 MAIN    9
C                                                                        MAIN   10
      COMMON NP,NT,NR,NI,NDEL,ISUB                                       MAIN   11
C                                                                        MAIN   12
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                    MAIN   13
      COMMON DIRCOS                                                      MAIN   14
      COMMON TIME                                                        MAIN   15
      COMMON IRARF                                                       MAIN   16
      COMMON KSTOP                                                       MAIN   17
      COMMON TPSI                                                        MAIN   18
      COMMON KKK                                                         MAIN   19
C                                                                        MAIN   20
C                                                                        MAIN   21
      REAL LO,MO,LENGTH,MU,KO                                            MAIN   22
      KR=10                                                              MAIN   23
      EPS=.0000001                                                       MAIN   24
      KSTOP=0                                                            MAIN   25
1     FORMAT (1H1)                                                       MAIN   26
      CALL DVCHK(KEY)                                                    MAIN   27
      KICK=0                                                             MAIN   28
      IF (KEY.EQ.1) GO TO 9980                                           MAIN   29
C                                                                        MAIN   30
      DO 2 K=1,6                                                         MAIN   31
      DO 2 J=1,20                                                        MAIN   32
      DO 2 I=1,20                                                        MAIN   33
      XMESH(I,J,K)=0.                                                    MAIN   34
      XMESH2(I,J,K)=0.                                                   MAIN   35
2     CONTINUE                                                           MAIN   36
```

```
C       KRW=0                                                                  MAIN  37
        NUZON=0                                                                MAIN  38
C                                                                              MAIN  39
C                                                                              MAIN  40
        WRITE (3,4)                                                            MAIN  41
C                                                                              MAIN  42
C           DATA INPUT SECTION                                                 MAIN  43
C                                                                              MAIN  44
4       FORMAT (52H1HYPERVELOCITY IMPACT METHOD OF CHARACTERISTICS CODE///MAIN  45
       1)                                                                      MAIN  46
C         ID AND FX. PT. CONSTANTS                                             MAIN  47
C                                                                              MAIN  48
        READ (1,8) CASEID,ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,NDEL        MAIN  49
8       FORMAT (13A6,A2/9I3)                                                   MAIN  50
        IRARF=ITI2                                                             MAIN  51
C                                                                              MAIN  52
C         FL. PT. CONSTANTS                                                    MAIN  53
C                                                                              MAIN  54
        READ (1,15) EPS1,EPS2,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5MAIN  55
       1,EPI6,EPI7                                                             MAIN  56
        READ (1,14) NP,NT,NR                                                   MAIN  57
14      FORMAT (3I3)                                                           MAIN  58
        READ (1,15) APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,REFMAIN  59
       1L                                                                      MAIN  60
        READ (1,15) ZMIN,ZMAX,RMIN,RMAX,GR,GZ,DELTA,VP,LENGTH,RADIUS,HSTARMAIN  61
15      FORMAT (6E12.8)                                                        MAIN  62
        READ (1,15) RST                                                        MAIN  63
        IF (RST.GT.0.) GO TO 16                                                MAIN  64
        REWIND 9                                                               MAIN  65
C       DO 1529 JTP=1,200                                                      MAIN  66
        READ (9)TIME                                                           MAIN  67
C       IF(ABS(RST+TIME).LT..001) GO TO 1530                                   MAIN  68
C       READ(9)BLOB                                                            MAIN  69
C1529 CONTINUE                                                                 MAIN  70
1530    CONTINUE                                                               MAIN  71
C       KRW=1                                                                  MAIN  72
```

```
      READ (9)(((XMESH(I,J,K),I=1,20),J=1,20),K=1,6),(Z(I),I=1,20),(R(I)MAIN  73
     1,I=1,20),((SURF(I,J),I=1,15),J=1,8),(((TAB(I,J,K),I=1,15),J=1,14),MAIN  74
     1K=1,2),((RARF(I,J),I=1,15),J=1,11),TIME,ZMIN,ZMAX,RMIN,RMAX,GR,GZ,MAIN  75
     1AR                                                                  MAIN  76
      DO 1500 J=1,20                                                      MAIN  77
      DO 1500 I=1,20                                                      MAIN  78
      DO 1500 K=1,6                                                       MAIN  79
1500  XMESH2(I,J,K)=XMESH(I,J,K)                                          MAIN  80
      DO 1501 I=1,15                                                      MAIN  81
      DO 1501 J=1,8                                                       MAIN  82
1501  SURF2(I,J)=SURF(I,J)                                                MAIN  83
      WRITE (3,145) TIME                                                  MAIN  84
      CALL SOUT                                                           MAIN  85
      CALL PRINT(XMESH2,Z,R,1)                                            MAIN  86
      KREFL=0                                                             MAIN  87
16    CONTINUE                                                            MAIN  88
      WRITE (3,10) CASEID,ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,NDEL   MAIN  89
10    FORMAT (1X13A6,A2//17H SHOCK ITERATIONS6X,4I4//20H INTERIOR ITERATMAIN  90
     1IONS3X,4I4//7H NDEL =,I4///)                                        MAIN  91
      WRITE (3,18) EPS1,EPS2,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPIMAIN  92
     15,EPI6,EPI7,ZMIN,ZMAX,RMIN,RMAX,DELTA,VP,LENGTH,RADIUS,APR,BPR,BIGMAIN  93
     1APR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,REFL                        MAIN  94
18    FORMAT (///38H ERROR CRITERIA FOR SHOCK COMPUTATIONS//5X8HDELTA Z1MAIN  95
     18X8HDELTA R18X9HDELTA RHO7X7HDELTA E9X7HDELTA P9X7HDELTA U/6E16.6/MAIN  96
     1//41H ERROR CRITERIA FOR INTERIOR COMPUTATIONS//5X8HDELTA Z18X8HDEMAIN  97
     1LTA R18X7HDELTA P9X7HDELTA U9X7HDELTA V9X9HDELTA RHO7X7HDELTA E/7EMAIN  98
     116.6///5X4HZMIN12X4HZMAX12X4HRMIN12X4HRMAX12X5HDELTA11X2HVP14X6HLEMAIN  99
     1NGTH10X6HRADIUS/8E16.6///5X2HA'14X2HB'14X6HBIG A'10X6HBIG B'10X2HEMAIN 100
     1*14X5HALPHA11X4HBETA12X4HRHO*/8E16.6//5X3HE'S13X4HREFL/2E16.6///) MAIN 101
      IF (RST.LT.0.) GO TO 140                                            MAIN 102
C                                                                         MAIN 103
C          STORE RHO* IN ALL XMESH                                        MAIN 104
C                                                                         MAIN 105
      DO 22 J=1,20                                                        MAIN 106
      DO 22 I=1,20                                                        MAIN 107
22    XMESH(I,J,4)=RHOSTR                                                 MAIN 108
```

```
40    FORMAT (5E12.8)                                                    MAIN 109
C                                                                        MAIN 110
C       PROJECTILE SHOCK                                                 MAIN 111
C                                                                        MAIN 112
      CALL EQOS1(PRHO,PPP,PVV,PEE,TEE,TRHO,KICK)                         MAIN 113
      IF (KICK.EQ.2200) GO TO 9980                                       MAIN 114
      DO 230 N=1,NP                                                      MAIN 115
      TAB(N,1,1)=(PRHO*PVV-RHOSTR*VP)*HSTAR/(PRHO-RHOSTR)                MAIN 116
      EE=N-1                                                             MAIN 117
      FNP=NP                                                             MAIN 118
      TAB(N,2,1)=RMIN+EE*(RADIUS-RMIN)/FNP                               MAIN 119
      TAB(N,3,1)=PPP                                                     MAIN 120
      TAB(N,4,1)=0.0                                                     MAIN 121
      TAB(N,5,1)=PVV                                                     MAIN 122
      TAB(N,6,1)=PRHO                                                    MAIN 123
      TAB(N,7,1)=PEE                                                     MAIN 124
      TAB(N,9,1)=0.0                                                     MAIN 125
      TAB(N,10,1)=1.0                                                    MAIN 126
230   CONTINUE                                                           MAIN 127
C                                                                        MAIN 128
C       TARGET SHOCK                                                     MAIN 129
C                                                                        MAIN 130
      M=0                                                                MAIN 131
      DO 240 N=1,NT                                                      MAIN 132
      EE=N-1                                                             MAIN 133
      FNT=NT-4                                                           MAIN 134
      TAB(N,2,2)=RMIN+EE*(RADIUS-RMIN)/FNT                               MAIN 135
      TAB(N,7,2)=TEE                                                     MAIN 136
      TAB(N,6,2)=TRHO                                                    MAIN 137
      TAB(N,3,2)=PPP                                                     MAIN 138
      IF (TAB(N,2,2).GT.RADIUS) GO TO 250                                MAIN 139
      TAB(N,1,2)=TRHO*PVV*HSTAR/(TRHO-RHOSTR)                            MAIN 140
      TAB(N,4,2)=0.0                                                     MAIN 141
      TAB(N,5,2)=PVV                                                     MAIN 142
      TAB(N,9,2)=0.0                                                     MAIN 143
      TAB(N,10,2)=1.0                                                    MAIN 144
```

```
      GO TO 240                                                          MAIN 145
250   EF=M                                                               MAIN 146
      TAB(N,9,2)=SIN(.5236+EF*.2618)                                     MAIN 147
      TAB(N,10,2)=COS(.5236+EF*.2618)                                    MAIN 148
      TAB(N,4,2)=PVV*TAB(N,9,2)                                          MAIN 149
      TAB(N,5,2)=PVV*TAB(N,10,2)                                         MAIN 150
      TAB(N,1,2)=TAB(1,1,2)*TAB(N,10,2)                                  MAIN 151
      TAB(N,2,2)=RADIUS+TAB(1,1,2)*TAB(N,9,2)                            MAIN 152
      M=M+1                                                              MAIN 153
240   CONTINUE                                                           MAIN 154
C                                                                        MAIN 155
C       RAREFACTION                                                      MAIN 156
C                                                                        MAIN 157
      CALL EQOS2(PPP,PRHO,PEE)                                           MAIN 158
      EE=NR-1                                                            MAIN 159
      ADEL=(TAB(1,1,2)-TAB(1,1,1))/EE                                    MAIN 160
      RARF(1,1)=TAB(1,1,2)                                               MAIN 161
      DO 205 N=2,NR                                                      MAIN 162
      RARF(N,1)=RARF(N-1,1)-ADEL                                         MAIN 163
205   CONTINUE                                                           MAIN 164
      DO 210 N=1,NR                                                      MAIN 165
      RARF(N,10)=(RARF(N,1)/HSTAR-.5*VP)/AR                              MAIN 166
      RARF(N,9)=-SQRT(1.-RARF(N,10)**2)                                  MAIN 167
      RARF(N,2)=RADIUS+HSTAR*AR*RARF(N,9)                                MAIN 168
210   CONTINUE                                                           MAIN 169
      DO 220 N=1,NR                                                      MAIN 170
      RARF(N,3)=TAB(1,3,1)                                               MAIN 171
      RARF(N,4)=0.                                                       MAIN 172
      RARF(N,5)=TAB(1,5,1)                                               MAIN 173
      RARF(N,6)=TAB(1,6,1)                                               MAIN 174
      RARF(N,7)=TAB(1,7,1)                                               MAIN 175
220   CONTINUE                                                           MAIN 176
C       REGION INTERIOR TO SHOCKS                                        MAIN 177
      I=-ZMIN/GZ+1.2                                                     MAIN 178
      J=0                                                                MAIN 179
260   J=J+1                                                              MAIN 180
```

```
      XMESH(I,J,1)=PPP                                                  MAIN 181
      XMESH(I,J,3)=PVV                                                  MAIN 182
      XMESH(I,J,4)=PRHO                                                 MAIN 183
      XMESH(I,J,5)=PEE                                                  MAIN 184
      EE=(J-1)                                                          MAIN 185
      IF ((EE*GR-RADIUS).LT.-EPS) GO TO 260                             MAIN 186
      XMESH(I,J,1)=0.                                                   MAIN 187
      XMESH(I,J,4)=RHOSTR                                               MAIN 188
      XMESH(I,J,5)=0.                                                   MAIN 189
C                                                                       MAIN 190
C       FREE SURFACE                                                    MAIN 191
C                                                                       MAIN 192
      DO 50 I=1,NP                                                      MAIN 193
      SURF(I,1)=-LENGTH+VP*HSTAR                                        MAIN 194
      SURF(I,2)=TAB(I,2,1)                                              MAIN 195
      SURF(I,3)=0.                                                      MAIN 196
      SURF(I,4)=0.                                                      MAIN 197
      SURF(I,5)=VP                                                      MAIN 198
      SURF(I,6)=RHOSTR                                                  MAIN 199
      SURF(I,7)=0.                                                      MAIN 200
      SURF(I,8)=SQRT(BIGAPR/RHOSTR)                                     MAIN 201
50    CONTINUE                                                          MAIN 202
      DO 51 I=1,NP                                                      MAIN 203
      DO 51 J=1,8                                                       MAIN 204
51    SURF2(I,J)=SURF(I,J)                                              MAIN 205
      IF (NUZON.EQ.0) GO TO 5001                                        MAIN 206
5000  GR=GR*2.                                                          MAIN 207
      GZ=GZ*2.                                                          MAIN 208
      ZMAX=ZMAX*2.-ZMIN                                                 MAIN 209
      RMAX=RMAX*2.                                                      MAIN 210
      NUZON=1                                                           MAIN 211
      WRITE (3,5003)                                                    MAIN 212
5003  FORMAT (7H REZONE///)                                             MAIN 213
5001  CONTINUE                                                          MAIN 214
C                                                                       MAIN 215
C                                                                       MAIN 216
```

```
      DO 55 I=1,20                                                    MAIN 217
      EE=I-1                                                          MAIN 218
      Z(I)=ZMIN+EE*GZ                                                 MAIN 219
      R(I)=RMIN+EE*GR                                                 MAIN 220
55    CONTINUE                                                        MAIN 221
      IF (NUZON.EQ.0) GO TO 5101                                      MAIN 222
      DO 5100 I=1,10                                                  MAIN 223
      DO 5100 J=1,10                                                  MAIN 224
      DO 5100 K=1,6                                                   MAIN 225
      L=2*I-1                                                         MAIN 226
      M=2*J-1                                                         MAIN 227
      XMESH(I,J,K)=XMESH(L,M,K)                                       MAIN 228
5100  CONTINUE                                                        MAIN 229
      GO TO 157                                                       MAIN 230
5101  CONTINUE                                                        MAIN 231
C                                                                     MAIN 232
C          COMPUTE A FOR 2 SHOCKS AND MESH                            MAIN 233
C                                                                     MAIN 234
      DO 86 K=1,3                                                     MAIN 235
      GO TO (57,59,61),K                                              MAIN 236
57    NN=NP                                                           MAIN 237
      JJ=1                                                            MAIN 238
      GO TO 63                                                        MAIN 239
59    NN=NT                                                           MAIN 240
      JJ=1                                                            MAIN 241
      GO TO 63                                                        MAIN 242
61    NN=20                                                           MAIN 243
      JJ=20                                                           MAIN 244
63    DO 84 N=1,NN                                                    MAIN 245
      DO 82 J=1,JJ                                                    MAIN 246
      GO TO (65,65,68),K                                              MAIN 247
65    P=TAB(N,3,K)                                                    MAIN 248
      RHO=TAB(N,6,K)                                                  MAIN 249
      E=TAB(N,7,K)                                                    MAIN 250
      GO TO 70                                                        MAIN 251
68    P=XMESH(J,N,1)                                                  MAIN 252
```

```
      RHO=XMESH(J,N,4)                                                  MAIN 253
      E=XMESH(J,N,5)                                                    MAIN 254
70    CONTINUE                                                          MAIN 255
      CALL EQOS3(RHO,AA,E,P)                                            MAIN 256
      GO TO (76,76,78),K                                                MAIN 257
76    TAB(N,8,K)=AA                                                     MAIN 258
      GO TO 82                                                          MAIN 259
78    XMESH(J,N,6)=AA                                                   MAIN 260
82    CONTINUE                                                          MAIN 261
84    CONTINUE                                                          MAIN 262
86    CONTINUE                                                          MAIN 263
      KICK=86                                                           MAIN 264
      CALL DVCHK(KQ)                                                    MAIN 265
      IF (KQ.EQ.1) GO TO 9980                                           MAIN 266
C                                                                       MAIN 267
C          STORE A FOR RAREFACTION                                      MAIN 268
C                                                                       MAIN 269
      DO 90 I=1,NR                                                      MAIN 270
90    RARF(I,8)=AR                                                      MAIN 271
C                                                                       MAIN 272
C          COMPLETE SHOCK TABLES                                        MAIN 273
C                                                                       MAIN 274
      DO 99 K=1,2                                                       MAIN 275
      GO TO (92,94),K                                                   MAIN 276
92    NN=NP                                                             MAIN 277
      GO TO 95                                                          MAIN 278
94    NN=NT                                                             MAIN 279
      US=0.                                                             MAIN 280
95    DO 97 N=1,NN                                                      MAIN 281
      GO TO (93,96),K                                                   MAIN 282
93    CONTINUE                                                          MAIN 283
      US=VP*TAB(N,10,1)                                                 MAIN 284
96    CONTINUE                                                          MAIN 285
      TAB(N,11,K)=TAB(N,9,K)*TAB(N,4,K)+TAB(N,10,K)*TAB(N,5,K)          MAIN 286
      TAB(N,12,K)=TAB(N,9,K)*TAB(N,5,K)-TAB(N,10,K)*TAB(N,4,K)          MAIN 287
      TAB(N,13,K)=((TAB(N,6,K)*ABS(TAB(N,11,K)))/(TAB(N,6,K)-RHOSTR)-US)MAIN 288
```

```
     1*(-1.)**K                                                        MAIN 289
      TAB(N,14,K)=1.                                                   MAIN 290
97    CONTINUE                                                         MAIN 291
99    CONTINUE                                                         MAIN 292
C                                                                      MAIN 293
C                                                                      MAIN 294
C          STORE ALL XMESH IN XMESH2                                   MAIN 295
C                                                                      MAIN 296
130   DO 135 K=1,6                                                     MAIN 297
      DO 135 J=1,20                                                    MAIN 298
      DO 135 I=1,20                                                    MAIN 299
      XMESH2(I,J,K)=XMESH(I,J,K)                                       MAIN 300
135   CONTINUE                                                         MAIN 301
      TIME=HSTAR                                                       MAIN 302
      WRITE (3,145) TIME                                               MAIN 303
      CALL SOUT                                                        MAIN 304
      CALL PRINT(XMESH2,Z,R,1)                                         MAIN 305
C                                                                      MAIN 306
      KREFL=0                                                          MAIN 307
139   IF (KREFL.NE.0) GO TO 143                                        MAIN 308
C          ENTRY FOR TIME STEP                                         MAIN 309
C                                                                      MAIN 310
140   READ (1,142) H                                                   MAIN 311
142   FORMAT (E12.8)                                                   MAIN 312
143   CONTINUE                                                         MAIN 313
      TIME=TIME+H                                                      MAIN 314
      WRITE (3,145) TIME                                               MAIN 315
145   FORMAT (1H1///6H TIME=,E15.8///)                                 MAIN 316
      WRITE (3,999) KR                                                 MAIN 317
999   FORMAT (5X,4H KR=,I5)                                            MAIN 318
C                                                                      MAIN 319
C                                                                      MAIN 320
C          ADVANCE SHOCK POINTS                                        MAIN 321
C                                                                      MAIN 322
      DO 159 N=1,NP                                                    MAIN 323
      IF (TAB(N,14,1).LT.0.) GO TO 156                                 MAIN 324
```

```
      IF ((TAB(N,1,1)-SURF(N,1)).GT.EPS) GO TO 154                          MAIN 325
156   TAB2(N,1,1)=TAB(N,1,1)                                                MAIN 326
      TAB2(N,2,1)=TAB(N,2,1)                                                MAIN 327
      TAB(N,14,1)=-1.                                                       MAIN 328
      GO TO 159                                                             MAIN 329
154   TAB2(N,1,1)=TAB(N,1,1)+TAB(N,13,1)*H*TAB(N,10,1)-VP*TAB(N,9,1)*TABMAIN 330
     1(N,10,1)*H                                                            MAIN 331
150   TAB2(N,2,1)=TAB(N,2,1)+TAB(N,13,1)*H*TAB(N,9,1)-VP*TAB(N,9,1)**2*HMAIN 332
159   CONTINUE                                                              MAIN 333
      DO 155 N=1,NT                                                         MAIN 334
      TAB2(N,1,2)=TAB(N,1,2)+TAB(N,13,2)*H*TAB(N,10,2)                      MAIN 335
155   TAB2(N,2,2)=TAB(N,2,2)+TAB(N,13,2)*H*TAB(N,9,2)                       MAIN 336
      DO 158 M=1,NT                                                         MAIN 337
      IF (TAB2(M,1,2).GT.ZMAX) GO TO 5000                                   MAIN 338
      IF (TAB2(M,2,2).GT.RMAX) GO TO 5000                                   MAIN 339
158   CONTINUE                                                              MAIN 340
157   NUZON=0                                                               MAIN 341
C          ADVANCE RAREFACTION                                              MAIN 342
C                                                                           MAIN 343
      IF (RARF(1,2).LT.0.)IRARF=1                                           MAIN 344
      IF (IRARF.EQ.1) GO TO 516                                             MAIN 345
      ENR=NR-1                                                              MAIN 346
      ADEL=(TAB2(1,1,2)-TAB2(1,1,1))/ENR                                    MAIN 347
      RARF2(1,1)=TAB2(1,1,2)                                                MAIN 348
      DO 510 N=2,NR                                                         MAIN 349
      RARF2(N,1)=RARF2(N-1,1)-ADEL                                          MAIN 350
510   CONTINUE                                                              MAIN 351
      DO 515 N=1,NR                                                         MAIN 352
      RARF2(N,3)=(RARF2(N,1)/TIME-.5*VP)/AR                                 MAIN 353
      RARF2(N,4)=-SQRT(1.-RARF2(N,3)**2)                                    MAIN 354
      RARF2(N,2)=RADIUS+TIME*AR*RARF2(N,4)                                  MAIN 355
515   CONTINUE                                                              MAIN 356
516   CONTINUE                                                              MAIN 357
      CALL SHOCK                                                            MAIN 358
C                                                                           MAIN 359
C                                                                           MAIN 360
```

```
      IF (ITS3.EQ.1) GO TO 569                                          MAIN 361
C          SHOCK COMPUTATIONS COMPLETED                                 MAIN 362
C          COMPUTE PARTICLE CURVES                                      MAIN 363
C                                                                       MAIN 364
      TMP=.5*VP*H                                                       MAIN 365
      DO 520 N=1,NP                                                     MAIN 366
      SPART(N,1,1)=TAB(N,1,1)+TAB(N,5,1)*H                              MAIN 367
520   SPART(N,2,1)=TAB(N,2,1)+TAB(N,4,1)*H                              MAIN 368
      DO 525 N=1,NT                                                     MAIN 369
      SPART(N,1,2)=TAB(N,1,2)+TAB(N,5,2)*H                              MAIN 370
525   SPART(N,2,2)=TAB(N,2,2)+TAB(N,4,2)*H                              MAIN 371
      IF (IRARF.EQ.1) GO TO 531                                         MAIN 372
      DO 530 N=1,NR                                                     MAIN 373
      RPART(N,1)=RARF(N,1)+TMP                                          MAIN 374
530   RPART(N,2)=RARF(N,2)                                              MAIN 375
531   CONTINUE                                                          MAIN 376
C                                                                       MAIN 377
C                                                                       MAIN 378
568   CALL SOUT2                                                        MAIN 379
569   CONTINUE                                                          MAIN 380
C       ADVANCE PROJECTILE REAR SURFACE                                 MAIN 381
      DO 5300 I=1,NP                                                    MAIN 382
      DO 5300 J=1,8                                                     MAIN 383
5300  SURF2(I,J)=SURF(I,J)                                              MAIN 384
      KICK=568                                                          MAIN 385
      CALL DVCHK(KQ)                                                    MAIN 386
      IF (KQ.EQ.1) GO TO 9980                                           MAIN 387
C                                                                       MAIN 388
C          START INTERIOR REGION COMPUTATIONS                           MAIN 389
C                                                                       MAIN 390
      CALL INTER                                                        MAIN 391
C                                                                       MAIN 392
C          INTERIOR REGION COMPUTATIONS COMPLETED                       MAIN 393
C                                                                       MAIN 394
      CALL PRINT(XMESH2,Z,R,2)                                          MAIN 395
570   CONTINUE                                                          MAIN 396
```

```
C                                                                       MAIN 397
C          INITIALIZE FOR NEXT TIME STEP                                MAIN 398
C                                                                       MAIN 399
      DO 920 K=1,6                                                      MAIN 400
      DO 920 J=1,20                                                     MAIN 401
      DO 920 I=1,20                                                     MAIN 402
      XMESH(I,J,K)=XMESH2(I,J,K)                                        MAIN 403
920   CONTINUE                                                          MAIN 404
      DO 930 J=1,13                                                     MAIN 405
      DO 930 I=1,NP                                                     MAIN 406
      TAB(I,J,1)=TAB2(I,J,1)                                            MAIN 407
930   CONTINUE                                                          MAIN 408
      DO 940 J=1,13                                                     MAIN 409
      DO 940 I=1,NT                                                     MAIN 410
      TAB(I,J,2)=TAB2(I,J,2)                                            MAIN 411
940   CONTINUE                                                          MAIN 412
      IF (IRARF.EQ.1) GO TO 951                                         MAIN 413
      DO 950 I=1,NR                                                     MAIN 414
      RARF(I,1)=RARF2(I,1)                                              MAIN 415
      RARF(I,2)=RARF2(I,2)                                              MAIN 416
950   CONTINUE                                                          MAIN 417
951   CONTINUE                                                          MAIN 418
      DO 960 I=1,NP                                                     MAIN 419
      DO 959 J=1,5                                                      MAIN 420
      SURF(I,J)=SURF2(I,J)                                              MAIN 421
959   CONTINUE                                                          MAIN 422
960   CONTINUE                                                          MAIN 423
      IF (KR.EQ.10) GO TO 888                                           MAIN 424
      REWIND 4                                                          MAIN 425
      WRITE (4)TIME                                                     MAIN 426
      WRITE (4)(((XMESH(I,J,K),I=1,20),J=1,20),K=1,6),(Z(I),I=1,20),(R(IMAIN 427
     1),I=1,20),((SURF(I,J),I=1,15),J=1,8),(((TAB(I,J,K),I=1,15),J=1,14)MAIN 428
     1,K=1,2),((RARF(I,J),I=1,15),J=1,11),TIME,ZMIN,ZMAX,RMIN,RMAX,GR,GZMAIN 429
     1,AR                                                               MAIN 430
      KR=10                                                             MAIN 431
      GO TO 777                                                         MAIN 432
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888   REWIND 10                                                          MAIN 433
C     IF(KRW.EQ.1)REWIND10                                               MAIN 434
C     KRW=0                                                              MAIN 435
      WRITE (10)TIME                                                     MAIN 436
      WRITE (10)(((XMESH(I,J,K),I=1,20),J=1,20),K=1,6),(Z(I),I=1,20),(R(MAIN 437
     1I),I=1,20),((SURF(I,J),I=1,15),J=1,8),(((TAB(I,J,K),I=1,15),J=1,14MAIN 438
     1),K=1,2),((RARF(I,J),I=1,15),J=1,11),TIME,ZMIN,ZMAX,RMIN,RMAX,GR,GMAIN 439
     1Z,AR                                                               MAIN 440
      KR=4                                                               MAIN 441
777   IF (KREFL.EQ.0) GO TO 980                                          MAIN 442
      KREFL=0                                                            MAIN 443
      H=H1                                                               MAIN 444
      GO TO 143                                                          MAIN 445
980   CONTINUE                                                           MAIN 446
      CALL DVCHK(KICK)                                                   MAIN 447
      IF (KICK.EQ.2) GO TO 140                                           MAIN 448
      WRITE (3,970)                                                      MAIN 449
970   FORMAT (28HODIVIDE CHECK AT END OF CASE/1H1)                       MAIN 450
      CALL EXIT                                                          MAIN 451
C                                                                        MAIN 452
C         DIVIDE CHECK                                                   MAIN 453
C                                                                        MAIN 454
9980  WRITE (3,9985) KICK                                                MAIN 455
9985  FORMAT (32HODIVIDE CHECK NEAR STATEMENT NO.,I5/1H1)                MAIN 456
      RETURN                                                             MAIN 457
      END                                                                MAIN 458
      SUBROUTINE DBLTRP(ZX,RX,ANS)                                       SUB1   1
C                                                                        SUB1   2
C         1ST ORDER DOUBLE INTERPOLATION THAT CONSIDERS                  SUB1   3
C         LINES OF DISCONTINUITY IF IN CONSIDERED REGION                 SUB1   4
C                                                                        SUB1   5
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPSSUB1   6
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LENSUB1   7
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS       SUB1   8
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2SUB1   9
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(SUB1  10
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     115,4),RPART(15,2)                                                    SUB1   11
C                                                                          SUB1   12
C                                                                          SUB1   13
      COMMON ZO,RO,PO,UO,VO,LO,MO,RHOO,EO,AO,UBARO,VBARO                   SUB1   14
C                                                                          SUB1   15
      COMMON NP,NT,NR,NI,NDEL,ISUB                                         SUB1   16
C                                                                          SUB1   17
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                      SUB1   18
      COMMON DIRCOS                                                        SUB1   19
      COMMON TIME                                                          SUB1   20
      COMMON IRARF                                                         SUB1   21
      COMMON KSTOP                                                         SUB1   22
      COMMON TPSI                                                          SUB1   23
      COMMON KKK                                                           SUB1   24
      REAL LO,MO,LENGTH,MU,KO                                              SUB1   25
C                                                                          SUB1   26
      DIMENSION ANS(6),ANS1(2,8),ANS2(2,8),ZI(4),RI(4),IK(4)               SUB1   27
      CALL OVCHK(KEY)                                                      SUB1   28
      IF (KEY.EQ.2) GO TO 4                                                SUB1   29
      NO=0                                                                 SUB1   30
      GO TO 940                                                            SUB1   31
C                                                                          SUB1   32
C          FIND SUBSCRIPTS FOR GRID                                        SUB1   33
C                                                                          SUB1   34
4     I1=(ZX-ZMIN)/GZ+1.000001                                             SUB1   35
      I2=I1+1                                                              SUB1   36
      J1=(RX-RMIN)/GR+1.000001                                             SUB1   37
      J2=J1+1                                                              SUB1   38
      NN=NP                                                                SUB1   39
      IF (ITS3.EQ.1) GO TO 3                                               SUB1   40
      DO 1 K=1,2                                                           SUB1   41
      IF (K.EQ.2)NN=NT                                                     SUB1   42
      DO 1 I=1,NN                                                          SUB1   43
      ALF=SQRT((TAB(I,1,K)-ZX)**2+(TAB(I,2,K)-RX)**2)                      SUB1   44
      IF (ALF.GT.EPS1) GO TO 1                                             SUB1   45
      ANS(1)=TAB(I,3,K)                                                    SUB1   46
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      ANS(2)=TAB(I,4,K)                                                  SUB1   47
      ANS(3)=TAB(I,5,K)                                                  SUB1   48
      ANS(4)=TAB(I,6,K)                                                  SUB1   49
      ANS(5)=TAB(I,7,K)                                                  SUB1   50
      ANS(6)=TAB(I,8,K)                                                  SUB1   51
      RETURN                                                             SUB1   52
1     CONTINUE                                                           SUB1   53
      IF (IRARF.EQ.1) GO TO 3                                            SUB1   54
      DO 2 I=1,NR                                                        SUB1   55
      ALF=SQRT((RARF(I,1)-ZX)**2+(RARF(I,2)-RX)**2)                      SUB1   56
      IF (ALF.GT.EPI1) GO TO 2                                           SUB1   57
      ANS(1)=RARF(I,3)                                                   SUB1   58
      ANS(2)=RARF(I,4)                                                   SUB1   59
      ANS(3)=RARF(I,5)                                                   SUB1   60
      ANS(4)=RARF(I,6)                                                   SUB1   61
      ANS(5)=RARF(I,7)                                                   SUB1   62
      ANS(6)=RARF(I,8)                                                   SUB1   63
      RETURN                                                             SUB1   64
2     CONTINUE                                                           SUB1   65
3     CONTINUE                                                           SUB1   66
C                                                                        SUB1   67
C                                                                        SUB1   68
      ZXX=ZX+.01                                                         SUB1   69
      RXX=RX+.01                                                         SUB1   70
C                                                                        SUB1   71
C          I LOOP FOR UPPER AND LOWER Z GRID LINES                       SUB1   72
C                                                                        SUB1   73
      DO 800 I=1,2                                                       SUB1   74
      IF (ITS3.EQ.1) GO TO 14                                            SUB1   75
      IF (I.EQ.2) GO TO 8                                                SUB1   76
      II=I1                                                              SUB1   77
      GO TO 12                                                           SUB1   78
8     II=I2                                                              SUB1   79
12    M=COMP(ZX,RX,Z(II),R(J1))                                          SUB1   80
      IF (M.EQ.1) GO TO 13                                               SUB1   81
      MCOM=1                                                             SUB1   82
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      GO TO 20                                                          SUB1   83
13    M=COMP(ZX,RX,Z(II),R(J2))                                         SUB1   84
      IF (M.EQ.1) GO TO 14                                              SUB1   85
      MCOM=2                                                            SUB1   86
      GO TO 20                                                          SUB1   87
C                                                                       SUB1   88
C          GET 6 VALUES ON GRID LINES                                   SUB1   89
C                                                                       SUB1   90
14    DO 15 K=1,6                                                       SUB1   91
      ANS1(I,K+2)=XMESH(II,J1,K)+(XMESH(II,J2,K)-XMESH(II,J1,K))*(RX-R(JSUB1   92
     11))/(R(J2)-R(J1))                                                 SUB1   93
15    CONTINUE                                                          SUB1   94
      CALL DVCHK(NO)                                                    SUB1   95
      IF (NO.EQ.2) GO TO 17                                             SUB1   96
      NO=15                                                             SUB1   97
      GO TO 940                                                         SUB1   98
17    ANS1(I,1)=Z(II)                                                   SUB1   99
      ANS1(I,2)=RX                                                      SUB1  100
      GO TO 800                                                         SUB1  101
C                                                                       SUB1  102
C                                                                       SUB1  103
20    ZZ=Z(II)                                                          SUB1  104
      RR=RX                                                             SUB1  105
      M=COMP(ZX,RX,ZZ,RR)                                               SUB1  106
      IF (M.EQ.1) GO TO 300                                             SUB1  107
C                                                                       SUB1  108
C                                                                       SUB1  109
      DO 25 K=1,2                                                       SUB1  110
      KATCH=0                                                           SUB1  111
      GO TO (21,22),K                                                   SUB1  112
21    NN=NP                                                             SUB1  113
      GO TO 23                                                          SUB1  114
22    NN=NT                                                             SUB1  115
23    JJJ=NN-1                                                          SUB1  116
      DO 24 M=1,JJJ                                                     SUB1  117
      IF (M.EQ.JJJ) GO TO 210                                           SUB1  118
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      IF (RX.GT.TAB(M+1,2,K).OR.RX.LT.TAB(M,2,K)) GO TO 24                SUB1 119
210   ZI(K)=TAB(M+1,1,K)+(TAB(M,1,K)-TAB(M+1,1,K))*(RX-TAB(M+1,2,K))/(TASUB1 120
     1B(M,2,K)-TAB(M+1,2,K))                                              SUB1 121
      NO=210                                                              SUB1 122
      CALL DVCHK(KQ)                                                      SUB1 123
      IF (KQ.EQ.1) GO TO 940                                              SUB1 124
      IF (I.EQ.2) GO TO 211                                               SUB1 125
      IF (KATCH.EQ.1) GO TO 212                                           SUB1 126
      KATCH=1                                                             SUB1 127
      GO TO 213                                                           SUB1 128
212   IF ((ZX-ZI(K)).GT.(ZX-ZM)) GO TO 24                                 SUB1 129
      IF (ZI(K).GT.ZX) GO TO 24                                           SUB1 130
213   ZM=ZI(K)                                                            SUB1 131
      IF (K.EQ.2) GO TO 215                                               SUB1 132
      NPS=M                                                               SUB1 133
      GO TO 24                                                            SUB1 134
215   NTS=M                                                               SUB1 135
      GO TO 24                                                            SUB1 136
211   CONTINUE                                                            SUB1 137
      IF (KATCH.EQ.1) GO TO 26                                            SUB1 138
      KATCH=1                                                             SUB1 139
      GO TO 213                                                           SUB1 140
26    IF ((ZI(K)-ZX).GT.(ZM-ZX)) GO TO 24                                 SUB1 141
      IF (ZI(K).LT.ZX) GO TO 24                                           SUB1 142
      GO TO 213                                                           SUB1 143
24    CONTINUE                                                            SUB1 144
      ZI(K)=ZM                                                            SUB1 145
      IF (KATCH.NE.0) GO TO 25                                            SUB1 146
      ZI(K)=ZMAX+1.                                                       SUB1 147
25    CONTINUE                                                            SUB1 148
      IF (IRARF.EQ.1)ZI(3)=ZMAX+1.                                        SUB1 149
      IF (IRARF.EQ.1) GO TO 2504                                          SUB1 150
      KATCH=0                                                             SUB1 151
      JJJ=NR-1                                                            SUB1 152
      DO 27 M=1,JJJ                                                       SUB1 153
      ZI(3)=RARF(M+1,1)+(RARF(M,1)-RARF(M+1,1))*(RX-RARF(M+1,2))/(RARF(MSUB1 154
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     1,2)-RARF(M+1,2))                                                                  SUB1 155
      NO=25                                                                             SUB1 156
      CALL DVCHK(KQ)                                                                    SUB1 157
      IF (KQ.EQ.1) GO TO 940                                                            SUB1 158
      IF (ABS(ZI(3)-ZX).GT.1.E-5) GO TO 279                                             SUB1 159
      DO 2799 LN=1,6                                                                    SUB1 160
      LNN=LN+2                                                                          SUB1 161
      ANS(LN)=RARF(1,LNN)                                                               SUB1 162
2799  CONTINUE                                                                          SUB1 163
      GO TO 820                                                                         SUB1 164
279   CONTINUE                                                                          SUB1 165
      IF (I.EQ.2) GO TO 28                                                              SUB1 166
      IF (ZI(3).GT.ZX) GO TO 27                                                         SUB1 167
      IF (KATCH.EQ.1) GO TO 280                                                         SUB1 168
      KATCH=1                                                                           SUB1 169
      GO TO 281                                                                         SUB1 170
280   IF ((ZX-ZI(3)).GT.(ZX-ZM)) GO TO 27                                               SUB1 171
281   ZM=ZI(3)                                                                          SUB1 172
      MR=M                                                                              SUB1 173
      GO TO 27                                                                          SUB1 174
28    IF (ZI(3).LT.ZX) GO TO 27                                                         SUB1 175
      IF (KATCH.EQ.1) GO TO 282                                                         SUB1 176
      KATCH=1                                                                           SUB1 177
      GO TO 281                                                                         SUB1 178
282   IF ((ZI(3)-ZX).GT.(ZM-ZX)) GO TO 27                                               SUB1 179
      GO TO 281                                                                         SUB1 180
27    CONTINUE                                                                          SUB1 181
      ZI(3)=ZM                                                                          SUB1 182
2504  CONTINUE                                                                          SUB1 183
      KATCH=0                                                                           SUB1 184
      K=4                                                                               SUB1 185
      JJJ=NP-1                                                                          SUB1 186
      DO 2700 M=1,JJJ                                                                   SUB1 187
      IF (M.EQ.JJJ) GO TO 2710                                                          SUB1 188
      IF (RX.GT.SURF(M+1,2).OR.RX.LT.SURF(M,2)) GO TO 2700                              SUB1 189
2710  ZI(4)=SURF(M+1,1)+(SURF(M,1)-SURF(M+1,1))*(RX-SURF(M+1,2))/(SURF(MSUB1 190
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     1,2)-SURF(M+1,2))                                                  SUB1 191
      IF (KQ.EQ.1) GO TO 940                                            SUB1 192
      CALL DVCHK(KQ)                                                    SUB1 193
      NO=2710                                                           SUB1 194
      IF (I.EQ.2) GO TO 2711                                            SUB1 195
      IF (KATCH.EQ.1) GO TO 2712                                        SUB1 196
      KATCH=1                                                           SUB1 197
      GO TO 2713                                                        SUB1 198
2712  IF ((ZX-ZI(K)).GT.(ZX-ZM)) GO TO 2700                             SUB1 199
2713  ZM=ZI(4)                                                          SUB1 200
      MS=M                                                              SUB1 201
      GO TO 2700                                                        SUB1 202
2711  CONTINUE                                                          SUB1 203
      IF (KATCH.EQ.1) GO TO 2726                                        SUB1 204
      KATCH=1                                                           SUB1 205
      GO TO 2713                                                        SUB1 206
2726  IF ((ZI(K)-ZX).GT.(ZM-ZX)) GO TO 2700                             SUB1 207
      GO TO 2713                                                        SUB1 208
2700  CONTINUE                                                          SUB1 209
      ZI(4)=ZM                                                          SUB1 210
      IF (KATCH.NE.0) GO TO 2701                                        SUB1 211
      ZI(4)=ZMAX+1.                                                     SUB1 212
2701  CONTINUE                                                          SUB1 213
      RI(1)=RX                                                          SUB1 214
      RI(2)=RX                                                          SUB1 215
      RI(3)=RX                                                          SUB1 216
      RI(4)=RX                                                          SUB1 217
C         FIND INTERSECTION TO USE                                      SUB1 218
C                                                                       SUB1 219
30    KEY=0                                                             SUB1 220
      IF (I.EQ.2) GO TO 50                                              SUB1 221
C                                                                       SUB1 222
C         UPPER GRID LINE                                               SUB1 223
C                                                                       SUB1 224
      DO 40 KK=1,4                                                      SUB1 225
      IF (Z(II).GT.ZI(KK)) GO TO 40                                     SUB1 226
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      IF (ABS(ZI(KK)-ZX).LT.1.E-5) GO TO 35                              SUB1 227
      IF (ZI(KK).GT.ZX) GO TO 40                                         SUB1 228
      IF (KEY.EQ.0) GO TO 35                                             SUB1 229
      IF (ZI(KK).LE.ZI(KEEP)) GO TO 40                                   SUB1 230
      KEEP=KK                                                            SUB1 231
      GO TO 40                                                           SUB1 232
35    KEEP=KK                                                            SUB1 233
      KEY=1                                                              SUB1 234
40    CONTINUE                                                           SUB1 235
      GO TO 65                                                           SUB1 236
C                                                                        SUB1 237
C         LOWER GRID LINE                                                SUB1 238
C                                                                        SUB1 239
50    DO 60 KK=1,4                                                       SUB1 240
      IF (Z(II).LT.ZI(KK)) GO TO 60                                      SUB1 241
      IF (ABS(ZI(KK)-ZX).LT.1.E-5) GO TO 55                              SUB1 242
      IF (ZI(KK).LT.ZX) GO TO 60                                         SUB1 243
      IF (KEY.EQ.0) GO TO 55                                             SUB1 244
      IF (ZI(KK).GE.ZI(KEEP)) GO TO 60                                   SUB1 245
      KEEP=KK                                                            SUB1 246
      GO TO 60                                                           SUB1 247
55    KEEP=KK                                                            SUB1 248
      KEY=1                                                              SUB1 249
60    CONTINUE                                                           SUB1 250
C                                                                        SUB1 251
65    IF (KEY.NE.0) GO TO 70                                             SUB1 252
      WRITE (6,67) ZX,RX,I,(ZI(KEY),KEY=1,4),(RI(KEY),KEY=1,4)           SUB1 253
67    FORMAT (34HOERROR NEAR STATEMENT 65 IN DBLTRP/1X3HZX=,E15.8,4X3HRXSUB1 254
     1=,F15.8,4X2HI=,I3/1X3HZI=,4E20.8/1X3HRI=,4E20.8/1H1)               SUB1 255
      XYZ=-2.                                                            SUB1 256
      ZYX=SQRT(XYZ)                                                      SUB1 257
      CALL EXIT                                                          SUB1 258
C                                                                        SUB1 259
C         FIND 6 VALUES ON SELECTED DISCONTINUITY                        SUB1 260
C                                                                        SUB1 261
70    IF (KEEP.EQ.3) GO TO 80                                            SUB1 262
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      IF (KEEP.EQ.4) GO TO 81                                                SUB1 263
      IF (KEEP.EQ.2) GO TO 71                                                SUB1 264
      N=NPS                                                                  SUB1 265
      GO TO 72                                                               SUB1 266
71    N=NTS                                                                  SUB1 267
72    CONTINUE                                                               SUB1 268
      ZY=ZI(KEEP)                                                            SUB1 269
      RY=RX                                                                  SUB1 270
      DO 75 K=3,8                                                            SUB1 271
      ANS1(I,K)=TAB(N,K,KEEP)+(TAB(N+1,K,KEEP)-TAB(N,K,KEEP))*SQRT(((RY-SUB1 272
     1TAB(N,2,KEEP))**2+(ZY-TAB(N,1,KEEP))**2)/((TAB(N+1,2,KEEP)-TAB(N,2SUB1 273
     1,KEEP))**2+(TAB(N+1,1,KEEP)-TAB(N,1,KEEP))**2))                       SUB1 274
      NO=75                                                                  SUB1 275
      CALL DVCHK(KQ)                                                         SUB1 276
      IF (KQ.EQ.1) GO TO 940                                                 SUB1 277
75    CONTINUE                                                               SUB1 278
      GO TO 90                                                               SUB1 279
80    N=MR                                                                   SUB1 280
      ZY=ZI(3)                                                               SUB1 281
      RY=RX                                                                  SUB1 282
      DO 85 K=3,8                                                            SUB1 283
      ANS1(I,K)=RARF(N,K)+(RARF(N+1,K)-RARF(N,K))*SQRT(((RY-RARF(N,2))**SUB1 284
     12+(ZY-RARF(N,1))**2)/((RARF(N+1,2)-RARF(N,2))**2+(RARF(N+1,1)-RARFSUB1 285
     1(N,1))**2))                                                            SUB1 286
      NO=85                                                                  SUB1 287
      CALL DVCHK(KQ)                                                         SUB1 288
      IF (KQ.EQ.1) GO TO 940                                                 SUB1 289
85    CONTINUE                                                               SUB1 290
      GO TO 90                                                               SUB1 291
81    N=MS                                                                   SUB1 292
      ZY=ZI(4)                                                               SUB1 293
      RY=RX                                                                  SUB1 294
      DO 86 K=3,8                                                            SUB1 295
      ANS1(I,K)=SURF(N,K)+(SURF(N+1,K)-SURF(N,K))*SQRT(((RY-SURF(N,2))**SUB1 296
     12+(ZY-SURF(N,1))**2)/((SURF(N+1,2)-SURF(N,2))**2+(SURF(N+1,1)-SURFSUB1 297
     1(N,1))**2))                                                            SUB1 298
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86    CONTINUE                                                          SUB1 299
90    CALL DVCHK(NO)                                                    SUB1 300
      IF (NO.EQ.2) GO TO 92                                             SUB1 301
      NO=90                                                             SUB1 302
      GO TO 940                                                         SUB1 303
92    ANSI(I,1)=ZY                                                      SUB1 304
      ANSI(I,2)=RY                                                      SUB1 305
      GO TO 800                                                         SUB1 306
C                                                                       SUB1 307
C          FIND INTERSECTIONS OF 3 DISCONTINUITIES AND Z GRID LINE      SUB1 308
C                                                                       SUB1 309
300   CONTINUE                                                          SUB1 310
      KATCHP=0                                                          SUB1 311
      KATCHR=0                                                          SUB1 312
      KATCHT=0                                                          SUB1 313
      KATCHS=0                                                          SUB1 314
      DO 310 K=1,2                                                      SUB1 315
      KATCH=0                                                           SUB1 316
      GO TO (303,301),K                                                 SUB1 317
303   NN=NP                                                             SUB1 318
      GO TO 302                                                         SUB1 319
301   NN=NT                                                             SUB1 320
302   JJJ=NN-1                                                          SUB1 321
      DO 309 M=1,JJJ                                                    SUB1 322
      IF ((TAB(M,1,K)-TAB(M+1,1,K)).GT.1.E-6) GO TO 3030                SUB1 323
      GO TO 309                                                         SUB1 324
3030  RI(K)=TAB(M+1,2,K)+(TAB(M,2,K)-TAB(M+1,2,K))*(Z(II)-TAB(M+1,1,K))/SUB1 325
     1(TAB(M,1,K)-TAB(M+1,1,K))                                         SUB1 326
      CALL DVCHK(NO)                                                    SUB1 327
      IF (NO.EQ.2) GO TO 3031                                           SUB1 328
      NO=3030                                                           SUB1 329
      GO TO 940                                                         SUB1 330
3031  CONTINUE                                                          SUB1 331
      IF (M.EQ.JJJ) GO TO 3022                                          SUB1 332
      IF (RI(K).GT.TAB(M+1,2,K).OR.RI(K).LT.TAB(M,2,K)) GO TO 309       SUB1 333
3022  IF (MCOM.EQ.2) GO TO 305                                          SUB1 334
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      IF (KATCH.EQ.1) GO TO 304                                          SUB1 335
      KATCH=1                                                            SUB1 336
      GO TO 3050                                                         SUB1 337
304   IF ((RX-RI(K)).GT.(RX-RM)) GO TO 309                               SUB1 338
3050  RM=RI(K)                                                           SUB1 339
      IF (K.EQ.2) GO TO 3040                                             SUB1 340
      MPS=M                                                              SUB1 341
      KATCHP=1                                                           SUB1 342
      GO TO 309                                                          SUB1 343
3040  MTS=M                                                              SUB1 344
      KATCHT=1                                                           SUB1 345
      GO TO 309                                                          SUB1 346
305   CONTINUE                                                           SUB1 347
      IF (KATCH.EQ.1) GO TO 306                                          SUB1 348
      KATCH=1                                                            SUB1 349
      GO TO 3050                                                         SUB1 350
306   IF ((RI(K)-RX).GT.(RM-RX)) GO TO 309                               SUB1 351
      GO TO 3050                                                         SUB1 352
309   CONTINUE                                                           SUB1 353
      RI(K)=RM                                                           SUB1 354
      IF (KATCH.NE.0) GO TO 310                                          SUB1 355
      RI(K)=RMAX+1.                                                      SUB1 356
310   CONTINUE                                                           SUB1 357
      K=3                                                                SUB1 358
      IF (IRARF.EQ.1)RI(3)=RMAX+1.                                       SUB1 359
      IF (IRARF.EQ.1) GO TO 315                                          SUB1 360
      JJJ=NR-1                                                           SUB1 361
      KATCH=0                                                            SUB1 362
      DO 312 M=1,JJJ                                                     SUB1 363
      RI(3)=RARF(M+1,2)+(RARF(M,2)-RARF(M+1,2))*(Z(II)-RARF(M+1,2))/(RARSUB1 364
     1F(M,1)-RARF(M+1,1))                                                SUB1 365
      NO=3122                                                            SUB1 366
      CALL DVCHK(KQ)                                                     SUB1 367
      IF (KQ.EQ.1) GO TO 940                                             SUB1 368
      IF (M.EQ.JJJ.OR.M.EQ.1) GO TO 3122                                 SUB1 369
      IF (RI(K).GT.RARF(M+1,2).OR.RI(K).LT.RARF(M,2)) GO TO 312          SUB1 370
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3122  IF (MCOM.EQ.2) GO TO 316                                          SUB1 371
      IF (KATCH.EQ.1) GO TO 317                                         SUB1 372
      KATCH=1                                                           SUB1 373
      GO TO 3051                                                        SUB1 374
317   IF ((RX-RI(K)).GT.(RX-RM)) GO TO 312                              SUB1 375
3051  RM=RI(K)                                                          SUB1 376
      MR=M                                                              SUB1 377
      KATCHR=1                                                          SUB1 378
      GO TO 312                                                         SUB1 379
316   CONTINUE                                                          SUB1 380
      IF (KATCH.EQ.1) GO TO 318                                         SUB1 381
      KATCH=1                                                           SUB1 382
      GO TO 3051                                                        SUB1 383
318   IF ((RI(K)-RX).GT.(RM-RX)) GO TO 312                              SUB1 384
      GO TO 3051                                                        SUB1 385
312   CONTINUE                                                          SUB1 386
      RI(K)=RM                                                          SUB1 387
      IF (KATCH.NE.0) GO TO 315                                         SUB1 388
      RI(K)=RMAX+1.                                                     SUB1 389
315   CONTINUE                                                          SUB1 390
      KATCH=0                                                           SUB1 391
      JJJ=NP-1                                                          SUB1 392
      DO 3150 M=1,JJJ                                                   SUB1 393
      IF ((SURF(M,1)-SURF(M+1,1)).GT.1.E-6) GO TO 3130                  SUB1 394
      GO TO 3150                                                        SUB1 395
3130  RI(4)=SURF(M+1,2)+(SURF(M,2)-SURF(M+1,2))*(Z(II)-SURF(M+1,1))/(SURSUB1 396
     1F(M,1)-SURF(M+1,1))                                               SUB1 397
      NO=3130                                                           SUB1 398
      CALL DVCHK(KQ)                                                    SUB1 399
      IF (KQ.EQ.1) GO TO 940                                            SUB1 400
      IF (M.EQ.JJJ) GO TO 3123                                          SUB1 401
      IF (RI(4).GT.SURF(M+1,2).OR.RI(4).LT.SURF(M,2)) GO TO 3150        SUB1 402
3123  IF (MCOM.EQ.2) GO TO 3105                                         SUB1 403
      IF (KATCH.EQ.1) GO TO 3104                                        SUB1 404
      KATCH=1                                                           SUB1 405
      GO TO 3109                                                        SUB1 406
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3104  IF ((RX-RI(4)).GT.(RX-RM)) GO TO 3150                          SUB1 407
3109  RM=RI(4)                                                       SUB1 408
      MS=M                                                           SUB1 409
      KATCHS=1                                                       SUB1 410
      GO TO 3150                                                     SUB1 411
3105  IF (KATCH.EQ.1) GO TO 3106                                     SUB1 412
      KATCH=1                                                        SUB1 413
      GO TO 3109                                                     SUB1 414
3106  IF ((RI(4)-RX).GT.(RM-RX)) GO TO 3150                          SUB1 415
      GO TO 3109                                                     SUB1 416
3150  CONTINUE                                                       SUB1 417
      RI(4)=RM                                                       SUB1 418
      IF (KATCH.NE.0) GO TO 3107                                     SUB1 419
      RI(4)=RMAX+1.                                                  SUB1 420
3107  CONTINUE                                                       SUB1 421
      IF (KATCHP+KATCHT+KATCHR+KATCHS.EQ.0) GO TO 485                SUB1 422
      ZI(1)=Z(II)                                                    SUB1 423
      ZI(2)=Z(II)                                                    SUB1 424
      ZI(3)=Z(II)                                                    SUB1 425
      ZI(4)=Z(II)                                                    SUB1 426
C         J LOOP FOR LEFT AND RIGHT R GRID LINES                     SUB1 427
C                                                                    SUB1 428
      DO 700 J=1,2                                                   SUB1 429
      IF (J.EQ.2) GO TO 350                                          SUB1 430
C                                                                    SUB1 431
C         LEFT R GRID LINE                                           SUB1 432
C                                                                    SUB1 433
      JJ=J1                                                          SUB1 434
      KEY=0                                                          SUB1 435
      DO 340 N=1,4                                                   SUB1 436
      IF (R(J1).GT.RI(N)) GO TO 340                                  SUB1 437
      IF (RI(N).GT.RX) GO TO 340                                     SUB1 438
      IF (KEY.EQ.1) GO TO 330                                        SUB1 439
      KEY=1                                                          SUB1 440
      KEEP=N                                                         SUB1 441
      GO TO 340                                                      SUB1 442
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C                                                                       SUB1 443
C          FIND CLOSEST                                                 SUB1 444
C                                                                       SUB1 445
330    DIF1=RX-RI(KEEP)                                                 SUB1 446
       DIF2=RX-RI(N)                                                    SUB1 447
       IF (DIF1.LE.DIF2) GO TO 340                                      SUB1 448
       KEEP=N                                                           SUB1 449
340    CONTINUE                                                         SUB1 450
       GO TO 375                                                        SUB1 451
C                                                                       SUB1 452
C          RIGHT R GRID LINE                                            SUB1 453
C                                                                       SUB1 454
350    JJ=J2                                                            SUB1 455
       KEY=0                                                            SUB1 456
       DO 360 N=1,4                                                     SUB1 457
       IF (R(J2).LT.RI(N)) GO TO 360                                    SUB1 458
       IF (RI(N).LT.RX) GO TO 360                                       SUB1 459
       IF (KEY.EQ.1) GO TO 355                                          SUB1 460
       KEY=1                                                            SUB1 461
       KEEP=N                                                           SUB1 462
       GO TO 360                                                        SUB1 463
355    DIF1=RI(KEEP)-RX                                                 SUB1 464
       DIF2=RI(N)-RX                                                    SUB1 465
       IF (DIF1.LE.DIF2) GO TO 360                                      SUB1 466
       KEEP=N                                                           SUB1 467
360    CONTINUE                                                         SUB1 468
375    IF (KEY.EQ.1) GO TO 400                                          SUB1 469
C                                                                       SUB1 470
C          NO POINTS BETWEEN RX AND GRID POINTS                         SUB1 471
C                                                                       SUB1 472
       ANS2(J,1)=Z(II)                                                  SUB1 473
       ANS2(J,2)=R(JJ)                                                  SUB1 474
       ANS2(J,3)=XMESH(II,JJ,1)                                         SUB1 475
       ANS2(J,4)=XMESH(II,JJ,2)                                         SUB1 476
       ANS2(J,5)=XMESH(II,JJ,3)                                         SUB1 477
       ANS2(J,6)=XMESH(II,JJ,4)                                         SUB1 478
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      ANS2(J,7)=XMESH(II,JJ,5)                                          SUB1 479
      ANS2(J,8)=XMESH(II,JJ,6)                                          SUB1 480
      GO TO 700                                                         SUB1 481
C                                                                       SUB1 482
C         POINT FOUND BETWEEN RX AND GRID POINTS                        SUB1 483
C                                                                       SUB1 484
400   GO TO (405,410,470,481),KEEP                                      SUB1 485
C                                                                       SUB1 486
C         INTERSECTION ON PROJECTILE SHOCK                              SUB1 487
C                                                                       SUB1 488
405   N1=MPS                                                            SUB1 489
      RY=RI(KEEP)                                                       SUB1 490
      ZY=Z(II)                                                          SUB1 491
      GO TO 520                                                         SUB1 492
C                                                                       SUB1 493
C         INTERSECTION ON TARGET SHOCK                                  SUB1 494
C                                                                       SUB1 495
410   N1=MTS                                                            SUB1 496
      RY=RI(KEEP)                                                       SUB1 497
      ZY=Z(II)                                                          SUB1 498
      GO TO 520                                                         SUB1 499
C                                                                       SUB1 500
C         INTERSECTION ON RAREFACTION                                   SUB1 501
C                                                                       SUB1 502
470   N1=MR                                                             SUB1 503
      RY=RI(KEEP)                                                       SUB1 504
      ZY=Z(II)                                                          SUB1 505
      GO TO 520                                                         SUB1 506
C                                                                       SUB1 507
C     INTERSECTION ON FREE SURFACE                                      SUB1 508
C                                                                       SUB1 509
481   N1=MS                                                             SUB1 510
      RY=RI(KEEP)                                                       SUB1 511
      ZY=Z(II)                                                          SUB1 512
      GO TO 520                                                         SUB1 513
C                                                                       SUB1 514
```

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C                                                                               SUB1  515
485   WRITE (3,488) KEEP,II,JJ,ZX,RX                                           SUB1  516
488   FORMAT (25HOERROR NEAR STATEMENT 485/1X5HKEEP=,I4,4X3HII=,I4,4X3HJSUB1  517
     1J=,I4/1X3HZX=,E15.8,4X3HRX=,E15.8/1H1)                                    SUB1  518
      CALL EXIT                                                                 SUB1  519
C                                                                               SUB1  520
C          FIND TABLE VALUES                                                    SUB1  521
C                                                                               SUB1  522
520   IF (KEEP.EQ.3) GO TO 580                                                  SUB1  523
      IF (KEEP.EQ.4) GO TO 591                                                  SUB1  524
      DO 550 N=3,8                                                              SUB1  525
      ANS2(J,N)=TAB(N1,N,KEEP)+(TAB(N1+1,N,KEEP)-TAB(N1,N,KEEP))*SQRT(((SUB1  526
     1RY-TAB(N1,2,KEEP))**2+(ZY-TAB(N1,1,KEEP))**2)/((TAB(N1+1,2,KEEP)-TSUB1  527
     1AB(N1,2,KEEP))**2+(TAB(N1+1,1,KEEP)-TAB(N1,1,KEEP))**2))                 SUB1  528
550   CONTINUE                                                                  SUB1  529
      IF (ZX.LT.0..AND.ABS(RX-RADIUS).LT.1.E-6) GO TO 552                       SUB1  530
      IF (RX.LT.RADIUS.OR.ABS(Z(II)).GT.1.E-6) GO TO 551                        SUB1  531
552   CONTINUE                                                                  SUB1  532
      ANS2(J,3)=0.                                                              SUB1  533
      ANS2(J,6)=RHOSTR                                                          SUB1  534
      ANS2(J,7)=0.                                                              SUB1  535
      ANS2(J,8)=SQRT(BIGAPR/RHOSTR)                                             SUB1  536
551   CONTINUE                                                                  SUB1  537
      GO TO 600                                                                 SUB1  538
580   DO 590 N=3,8                                                              SUB1  539
      ANS2(J,N)=RARF(N1,N)+(RARF(N1+1,N)-RARF(N1,N))*SQRT(((RY-RARF(N1,2SUB1  540
     11))**2+(ZY-RARF(N1,1))**2)/((RARF(N1+1,2)-RARF(N1,2))**2+(RARF(N1+1SUB1  541
     1,1)-RARF(N1,1))**2))                                                      SUB1  542
590   CONTINUE                                                                  SUB1  543
      GO TO 600                                                                 SUB1  544
591   DO 592 N=3,8                                                              SUB1  545
      ANS2(J,N)=SURF(N1,N)+(SURF(N1+1,N)-SURF(N1,N))*SQRT(((RY-SURF(N1,2SUB1  546
     11))**2+(ZY-SURF(N1,1))**2)/((SURF(N1+1,2)-SURF(N1,2))**2+(SURF(N1+1SUB1  547
     1,1)-SURF(N1,1))**2))                                                      SUB1  548
592   CONTINUE                                                                  SUB1  549
600   CALL DVCHK(NO)                                                            SUB1  550
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```
      IF (NO.EQ.2) GO TO 605                                                SUB1 551
      NO=600                                                                SUB1 552
      GO TO 940                                                             SUB1 553
605   ANS2(J,1)=ZY                                                          SUB1 554
      ANS2(J,2)=RY                                                          SUB1 555
C                                                                           SUB1 556
C         END OF LOOP FOR BOTH R GRID LINES                                 SUB1 557
C                                                                           SUB1 558
700   CONTINUE                                                              SUB1 559
C                                                                           SUB1 560
C         INTERPOLATE FOR UPPER AND LOWER VALUES                            SUB1 561
C                                                                           SUB1 562
      DO 720 J=3,8                                                          SUB1 563
      ANS1(I,J)=ANS2(1,J)+(ANS2(2,J)-ANS2(1,J))*(RX-ANS2(1,2))/(ANS2(2,2SUB1 564
     1)-ANS2(1,2))                                                          SUB1 565
720   CONTINUE                                                              SUB1 566
      CALL DVCHK(NO)                                                        SUB1 567
      IF (NO.EQ.2) GO TO 730                                                SUB1 568
      NO=720                                                                SUB1 569
      GO TO 940                                                             SUB1 570
730   ANS1(I,1)=Z(II)                                                       SUB1 571
      ANS1(I,2)=RX                                                          SUB1 572
C                                                                           SUB1 573
C         END OF LOOP FOR BOTH Z GRID LINES                                 SUB1 574
C                                                                           SUB1 575
800   CONTINUE                                                              SUB1 576
C                                                                           SUB1 577
C         FIND FINAL VALUES                                                 SUB1 578
C                                                                           SUB1 579
      DO 810 J=1,6                                                          SUB1 580
      ANS(J)=ANS1(1,J+2)+(ANS1(2,J+2)-ANS1(1,J+2))*(ZX-ANS1(1,1))/(ANS1(SUB1 581
     12,1)-ANS1(1,1))                                                       SUB1 582
810   CONTINUE                                                              SUB1 583
      CALL DVCHK(NO)                                                        SUB1 584
      IF (NO.EQ.2) GO TO 820                                                SUB1 585
      NO=810                                                                SUB1 586
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      GO TO 940                                                          SUB1  587
820   RETURN                                                             SUB1  588
C                                                                        SUB1  589
940   WRITE (3,942)                                                      SUB1  590
942   FORMAT (35HODIVIDE CHECK ERROR IN SUBR. DBLTRP)                    SUB1  591
950   WRITE (3,952) NO,ZX,RX,I1,J1,KEEP,ZI,RI                           SUB1  592
952   FORMAT (19H NEAR STATEMENT NO.,I4/1X3HZX=,E15.8,4X3HRX=,E15.8/1X3HSUB1  593
     1I1=,I4,4X3HJ1=,I4,4X5HKEEP=,I4/1X3HZI=,4E18.8/1X3HRI=,4E18.8)     SUB1  594
      WRITE (3,955) ((ANS1(I,J),J=1,8),I=1,2),((ANS2(I,J),J=1,8),I=1,2) SUB1  595
955   FORMAT (1X5HANS1=/8E16.8/8E16.8/1X5HANS2=/8E16.8/8E16.8/1H1)       SUB1  596
      XYZ=-2.                                                            SUB1  597
      ZYX=SQRT(XYZ)                                                      SUB1  598
      CALL EXIT                                                          SUB1  599
      RETURN                                                             SUB1  600
      END                                                                SUB1  601
      SUBROUTINE SHOCK                                                   SUB2    1
C         COMPUTES SHOCK VALUES                                          SUB2    2
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPSSUB2    3
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LENSUB2    4
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS       SUB2    5
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2SUB2    6
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(SUB2    7
     115,4),RPART(15,2)                                                 SUB2    8
C                                                                        SUB2    9
      COMMON ZO,RO,PO,UO,VO,LO,MO,RHOO,EO,AO,UBARO,VBARO                 SUB2   10
C                                                                        SUB2   11
      COMMON NP,NT,NR,NI,NDEL,ISUB                                       SUB2   12
C                                                                        SUB2   13
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                    SUB2   14
      COMMON DIRCOS                                                      SUB2   15
      COMMON TIME                                                        SUB2   16
      COMMON IRARF                                                       SUB2   17
      COMMON KSTOP                                                       SUB2   18
      COMMON TPSI                                                        SUB2   19
      COMMON KKK                                                         SUB2   20
C                                                                        SUB2   21
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      REAL LO,MO,LENGTH,MU,KO                                          SUB2  22
C                                                                      SUB2  23
      DIMENSION ANS(6)                                                 SUB2  24
      EXTERNAL FGOF1                                                   SUB2  25
      EPS=.0000001                                                     SUB2  26
C                                                                      SUB2  27
C         BEGIN SHOCK POINT COMPUTATIONS                               SUB2  28
      DO 505 K=1,2                                                     SUB2  29
      GO TO (158,160),K                                                SUB2  30
158   NN=NP                                                            SUB2  31
      GO TO 162                                                        SUB2  32
160   NN=NT                                                            SUB2  33
      VBARS=0.                                                         SUB2  34
162   DO 500 I=1,NN                                                    SUB2  35
      MPROJ=0                                                          SUB2  36
      IF (TAB(I,14,K).LT.0.) GO TO 500                                 SUB2  37
164   CONTINUE                                                         SUB2  38
C                                                                      SUB2  39
C         INITIALIZE TO ITERATE ON 1 SHOCK POINT                       SUB2  40
C                                                                      SUB2  41
      NBIC=0                                                           SUB2  42
      ZO=TAB2(I,1,K)                                                   SUB2  43
      RO=TAB2(I,2,K)                                                   SUB2  44
      PO=TAB(I,3,K)                                                    SUB2  45
      UO=TAB(I,4,K)                                                    SUB2  46
      VO=TAB(I,5,K)                                                    SUB2  47
      RHOO=TAB(I,6,K)                                                  SUB2  48
      EO=TAB(I,7,K)                                                    SUB2  49
      AO=TAB(I,8,K)                                                    SUB2  50
      LO=TAB(I,9,K)                                                    SUB2  51
      MO=TAB(I,10,K)                                                   SUB2  52
      UBARO=TAB(I,11,K)                                                SUB2  53
      VBARO=TAB(I,12,K)                                                SUB2  54
      UTOH=TAB(I,13,K)                                                 SUB2  55
      UTO=UTOH                                                         SUB2  56
      ITS44=ITS4                                                       SUB2  57
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      IF (IRARF.EQ.1) GO TO 170                                          SUB2  58
      M=1-(NR-2)*(K-2)                                                   SUB2  59
      FF=RO-RARF2(M+1,2)-(RARF2(M,2)-RARF2(M+1,2))*(ZO-RARF2(M+1,1))/(RASUB2  60
     1RF2(M,1)-RARF2(M+1,1))                                             SUB2  61
169   CONTINUE                                                           SUB2  62
      IF (FF.LT..001) GO TO 350                                          SUB2  63
170   IF (I.NE.1) GO TO 180                                              SUB2  64
      MO=TAB(I,10,K)                                                     SUB2  65
      LO=0.                                                              SUB2  66
      GO TO 190                                                          SUB2  67
180   IF (I.LT.NN) GO TO 184                                             SUB2  68
      UP=H*(TAB(I-1,13,K)-TAB(I,13,K))                                   SUB2  69
      TMP=SQRT((TAB2(I-1,2,K)-RO)**2+(TAB2(I-1,1,K)-ZO)**2)              SUB2  70
      DOMEG=UP/TMP                                                       SUB2  71
      GO TO 186                                                          SUB2  72
184   DR1=TAB2(I+1,2,K)-RO                                               SUB2  73
      DR2=RO-TAB2(I-1,2,K)                                               SUB2  74
      DZ1=TAB2(I+1,1,K)-ZO                                               SUB2  75
      DZ2=ZO-TAB2(I-1,1,K)                                               SUB2  76
      UP=H*(TAB(I-1,13,K)-TAB(I+1,13,K))                                 SUB2  77
      TMP=SQRT(DR1**2+DZ1**2)+SQRT(DR2**2+DZ2**2)                        SUB2  78
      DOMEG=UP/TMP                                                       SUB2  79
      KICK=184                                                           SUB2  80
      CALL DVCHK(KQ)                                                     SUB2  81
      IF (KQ.EQ.1) GO TO 9980                                            SUB2  82
C                                                                        SUB2  83
C         COMPUTE NEW LO,MO                                              SUB2  84
C                                                                        SUB2  85
186   COMEG=COS(DOMEG)                                                   SUB2  86
      SOMEG=SIN(DOMEG)                                                   SUB2  87
      XLO=LO*COMEG+MO*SOMEG                                              SUB2  88
      XMO=MO*COMEG-LO*SOMEG                                              SUB2  89
      LO=XLO                                                             SUB2  90
      MO=XMO                                                             SUB2  91
      IF (PO.GT..0025) GO TO 190                                         SUB2  92
      PO=0.                                                              SUB2  93
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      U0=0.                                                                  SUB2  94
      V0=VP*(1.-(-1.)**K)/2.                                                 SUB2  95
      RHO0=RHOSTR                                                            SUB2  96
      E0=0.                                                                  SUB2  97
      A0=SQRT(BIGAPR/RHOSTR)                                                 SUB2  98
      UBAR0=M0*V0                                                            SUB2  99
      VBAR0=VBARS                                                            SUB2 100
      UT0=UBAR0+A0*(-1.)**K                                                  SUB2 101
      GO TO 350                                                              SUB2 102
190   ITS33=ITS3                                                             SUB2 103
C                                                                            SUB2 104
C          FIND GUESS TO START ITERATION                                     SUB2 105
C                                                                            SUB2 106
195   CALL GUESS(1,KOD2,Z0,R0,I,K,ZZ,RR,DZ,DR)                               SUB2 107
      IF (KOD2.EQ.1) GO TO 200                                               SUB2 108
      WRITE (3,198) I,K,Z0,R0                                                SUB2 109
      WRITE (3,7002) ZZ,RR,DZ,DR                                             SUB2 110
198   FORMAT (31HONO GUESS FOUND FOR SHOCK POINT/3H0I=,I4,6X2HK=,I4,10X3SUB2 111
     1HZ0=,E15.8,10X3HR0=,E15.8/1H1)                                         SUB2 112
      CALL EXIT                                                              SUB2 113
200   CONTINUE                                                               SUB2 114
      KY=K                                                                   SUB2 115
      NTW=0                                                                  SUB2 116
      IF (K.EQ.2) GO TO 201                                                  SUB2 117
      VBARS=VP*L0                                                            SUB2 118
      DIRCOS=-M0                                                             SUB2 119
      GO TO 203                                                              SUB2 120
201   DIRCOS=M0                                                              SUB2 121
203   CONTINUE                                                               SUB2 122
      CALL NRIT2(Z1,R1,ZZ,DZ,RR,DR,EPS1,EPS2,FGOF1,ITS1,KODE)                SUB2 123
      IF (KODE.EQ.0) GO TO 205                                               SUB2 124
C                                                                            SUB2 125
C     BICHARACTERISTIC SELECTION SCHEME                                      SUB2 126
C                                                                            SUB2 127
      IF (NBIC.EQ.0) GO TO 204                                               SUB2 128
      IF (NTW.EQ.8)KY=KY+1                                                   SUB2 129
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      IF (NTW.GT.21) GO TO 7000                                          SUB2 130
      ANG1=ANG1+DTPSI*(-1.)**KY                                          SUB2 131
      DIRCOS=SIN(ANG1)                                                   SUB2 132
      LO=COS(ANG1)                                                       SUB2 133
      NTW=NTW+1                                                          SUB2 134
      GO TO 203                                                          SUB2 135
204   CONTINUE                                                           SUB2 136
      NBIC=1                                                             SUB2 137
      CALL DBLTRP(ZZ,RR,ANS)                                             SUB2 138
      UA=ANS(2)                                                          SUB2 139
      VA=ANS(3)                                                          SUB2 140
      AA=ANS(6)                                                          SUB2 141
      ZZZ=ZZ+DZ                                                          SUB2 142
      CALL DBLTRP(ZZZ,RR,ANS)                                            SUB2 143
      UB=ANS(2)                                                          SUB2 144
      VB=ANS(3)                                                          SUB2 145
      AB=ANS(6)                                                          SUB2 146
      RRR=RR+DR                                                          SUB2 147
      CALL DBLTRP(ZZ,RRR,ANS)                                            SUB2 148
      UC=ANS(2)                                                          SUB2 149
      VC=ANS(3)                                                          SUB2 150
      AC=ANS(6)                                                          SUB2 151
      MM=0                                                               SUB2 152
      TPSI=1.5708*(-1.)**K                                               SUB2 153
      XB=ZZ                                                              SUB2 154
      YB=RR                                                              SUB2 155
      NOM=5                                                              SUB2 156
      CA=NOM                                                             SUB2 157
6201  DTPSI=.01745                                                       SUB2 158
      DTPSI=CA*DTPSI                                                     SUB2 159
      TPSI=TPSI+DTPSI*(-1.)**K                                           SUB2 160
      A1=1.+H*(VB-VA+(AB-AA)*SIN(TPSI))/DZ                               SUB2 161
      B1=H*(VC-VA+(AC-AA)*SIN(TPSI))/DR                                  SUB2 162
      C1=-(ZZ-ZO+H*(VA+AA*SIN(TPSI)))                                    SUB2 163
      A2=H*(UB-UA+(AB-AA)*COS(TPSI))/DZ                                  SUB2 164
      B2=1.+H*(UC-UA+(AC-AA)*COS(TPSI))/DR                               SUB2 165
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      C2=-(RR-RO+H*(UA+AA*COS(TPSI)))                                    SUB2 166
      DET=A1*B2-A2*B1                                                    SUB2 167
      DELX=(B2*C1-B1*C2)/DET                                             SUB2 168
      DELY=(A1*C2-A2*C1)/DET                                             SUB2 169
C                                                                        SUB2 170
C          TEST FOR SAME REGION                                          SUB2 171
C                                                                        SUB2 172
      XB1=XB+DELX                                                        SUB2 173
      YB1=YB+DELY                                                        SUB2 174
      M=COMP(XB,YB,XB1,YB1)                                              SUB2 175
      IF (M.EQ.1) GO TO 6203                                             SUB2 176
      MM=MM+1                                                            SUB2 177
      IF (MM.LT.360/NOM) GO TO 6201                                      SUB2 178
7000  WRITE (3,7001)                                                     SUB2 179
7001  FORMAT (41HOBICHARACTERISTIC SELECTION SCHEME FAILED)              SUB2 180
      WRITE (3,614) ZO,RO                                                SUB2 181
614   FORMAT (1X5HZO  =,E15.8,4X5HRO  =,E15.8)                           SUB2 182
      WRITE (3,7002) UA,VA,AA,UB,VB,AB,UC,VC,AC,ZZ,DZ,RR,DR,ANG1         SUB2 183
7002  FORMAT (4E16.8)                                                    SUB2 184
      CALL EXIT                                                          SUB2 185
6203  CONTINUE                                                           SUB2 186
      WRITE (3,6210)                                                     SUB2 187
6210  FORMAT (53HOBICHARACTERISTIC SELECTION SCHEME EMPLOYED BY SHOCKX)  SUB2 188
6204  ANG1=TPSI                                                          SUB2 189
      T1=DIRCOS                                                          SUB2 190
      T2=LO                                                              SUB2 191
      DIRCOS=SIN(ANG1)                                                   SUB2 192
      LO=COS(ANG1)                                                       SUB2 193
      GO TO 203                                                          SUB2 194
205   CONTINUE                                                           SUB2 195
      UBARS1=0.                                                          SUB2 196
      IF (K.EQ.1)UBARS1=MO*VP                                            SUB2 197
      CALL DBLTRP(Z1,R1,ANS)                                             SUB2 198
      P1=ANS(1)                                                          SUB2 199
      U1=ANS(2)                                                          SUB2 200
      V1=ANS(3)                                                          SUB2 201
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      RHO1=ANS(4)                                                       SUB2 202
      E1=ANS(5)                                                         SUB2 203
      A1=ANS(6)                                                         SUB2 204
206   CONTINUE                                                          SUB2 205
      KICK=205                                                          SUB2 206
      CALL DVCHK(KQ)                                                    SUB2 207
      IF (KQ.EQ.1) GO TO 9980                                           SUB2 208
      IF (NBIC.EQ.0) GO TO 207                                          SUB2 209
7003  SINTH=ABS(DIRCOS)                                                 SUB2 210
      COSTH=ABS(LO)                                                     SUB2 211
      DIRCOS=T1                                                         SUB2 212
      LO=T2                                                             SUB2 213
207   CONTINUE                                                          SUB2 214
      UBAR1=LO*U1+MO*V1                                                 SUB2 215
      IF (K.EQ.2) GO TO 208                                             SUB2 216
      IF (UBAR1.LT.VP/2.) GO TO 208                                     SUB2 217
      UBAR1=UBARS1-UBAR1                                                SUB2 218
208   CONTINUE                                                          SUB2 219
218   CONTINUE                                                          SUB2 220
      M1=PART(1,Z1,R1,ZZ,RR,DELTA,NDEL)                                 SUB2 221
      IF (M1.EQ.1) GO TO 210                                            SUB2 222
      PUR=0.                                                            SUB2 223
      PVR=0.                                                            SUB2 224
      GO TO 215                                                         SUB2 225
210   CALL DBLTRP(ZZ,RR,ANS)                                            SUB2 226
      DP=RR-R1                                                          SUB2 227
      PUR=(ANS(2)-U1)/DP                                                SUB2 228
219   CONTINUE                                                          SUB2 229
      PVR=(ANS(3)-V1)/DP                                                SUB2 230
215   M1=PART(2,Z1,R1,ZZ,RR,DELTA,NDEL)                                 SUB2 231
      IF (M1.EQ.1) GO TO 220                                            SUB2 232
      PUZ=0.                                                            SUB2 233
      PVZ=0.                                                            SUB2 234
      PAZ=0.                                                            SUB2 235
      GO TO 225                                                         SUB2 236
220   CALL DBLTRP(ZZ,RR,ANS)                                            SUB2 237
```

```
      DP=ZZ-Z1                                                          SUB2 238
      PUZ=(ANS(2)-U1)/DP                                                SUB2 239
      PVZ=(ANS(3)-V1)/DP                                                SUB2 240
225   CONTINUE                                                          SUB2 241
      IF (NBIC.EQ.1) GO TO 7004                                         SUB2 242
      PURB1=LO*PUR+MO*PVR                                               SUB2 243
      PVRB1=LO*PVR-MO*PUR                                               SUB2 244
      PVZB1=LO*PVZ-MO*PUZ                                               SUB2 245
      PVEB1=-MO*PVRB1+LO*PVZB1                                          SUB2 246
      SBAR1=PVEB1                                                       SUB2 247
226   CONTINUE                                                          SUB2 248
      IF (ABS(R1).LE.EPS) GO TO 235                                     SUB2 249
      SBAR1=SBAR1+U1/R1                                                 SUB2 250
      GO TO 240                                                         SUB2 251
7004  CONTINUE                                                          SUB2 252
      IF (V1.GT.VP/2..AND.K.EQ.1)V1=VP-V1                               SUB2 253
      SBAR1=SINTH**2*PUR-SINTH*COSTH*(PUZ+PVR)+COSTH**2*PVZ             SUB2 254
      GO TO 226                                                         SUB2 255
235   SBAR1=SBAR1+PUR                                                   SUB2 256
C                                                                       SUB2 257
C                                                                       SUB2 258
240   ITS22=ITS2                                                        SUB2 259
      MMM=0                                                             SUB2 260
250   CONTINUE                                                          SUB2 261
      CALL EQOSS(PFR,PFE)                                               SUB2 262
      BIGA1=RHO1*A1                                                     SUB2 263
      KICK=250                                                          SUB2 264
      CALL DVCHK(KQ)                                                    SUB2 265
      IF (KQ.EQ.1) GO TO 9980                                           SUB2 266
      TEMP=1.-RHOSTR/RHOO                                               SUB2 267
      TMP=SQRT(PO*TEMP/RHOSTR)                                          SUB2 268
      IF (K.EQ.2) GO TO 251                                             SUB2 269
      IF (TMP.LT.VP/2.) GO TO 251                                       SUB2 270
      MPROJ=1                                                           SUB2 271
      TMP=UBARS1-TMP                                                    SUB2 272
251   CONTINUE                                                          SUB2 273
```

```
      TMP6=TMP                                                          SUB2 274
256   CONTINUE                                                          SUB2 275
      FNBIC=NBIC                                                        SUB2 276
      TMP1=P1+BIGA1*UBAR1-RHO1*H*SBAR1*A1**2+BIGA1*(-UBAR1+COSTH*U1+SINTSUB2 277
     1H*V1-SINTH*LO*VBARS+COSTH*MO*VBARS)*(FNBIC)                       SUB2 278
      TMP2=PFR*TEMP+PO*RHOSTR/RHOO**2                                   SUB2 279
      TMP5=PFE*TEMP                                                     SUB2 280
      GTMP=-RHO1*A1                                                     SUB2 281
      IF (NBIC.EQ.0) GO TO 259                                          SUB2 282
      GTMP=GTMP*(COSTH*LO+SINTH*MO)                                     SUB2 283
259   CONTINUE                                                          SUB2 284
      BIGG=PO-(GTMP*TMP+TMP1)                                           SUB2 285
      PGR=PFR-((GTMP*TMP2)/(2.*RHOSTR*TMP6))                            SUB2 286
      PGE=PFE-((GTMP*TMP5)/(2.*RHOSTR*TMP6))                            SUB2 287
265   TMP=.5*(1./RHOSTR-1./RHOO)                                        SUB2 288
      BIGH=EO-TMP*PO                                                    SUB2 289
      PHR=-TMP*PFR-.5*PO/RHOO**2                                        SUB2 290
      PHE=1.-TMP*PFE                                                    SUB2 291
      IF (ABS(BIGH).GT..0001) GO TO 267                                 SUB2 292
      BIGH=0.                                                           SUB2 293
267   IF (ABS(BIGG).GT..0001) GO TO 269                                 SUB2 294
      BIGG=0.                                                           SUB2 295
269   CONTINUE                                                          SUB2 296
C                                                                       SUB2 297
C          COMPUTE DELTA EO,DELTA RHOO                                  SUB2 298
C                                                                       SUB2 299
      DOWN=PGE*PHR-PGR*PHE                                              SUB2 300
      DEO=(-BIGG*PHR+BIGH*PGR)/DOWN                                     SUB2 301
      DRHOO=(-BIGH*PGE+BIGG*PHE)/DOWN                                   SUB2 302
C                                                                       SUB2 303
      EO2=EO+DEO                                                        SUB2 304
      IF (EO2.LT.0.)EO2=0.                                              SUB2 305
      RHOO2=RHOO+DRHOO                                                  SUB2 306
      IF (RHOO2.LT.RHOSTR)RHOO2=RHOO                                    SUB2 307
      KICK=265                                                          SUB2 308
      CALL DVCHK(KQ)                                                    SUB2 309
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```
      IF (KQ.EQ.1) GO TO 9980                                          SUB2 310
      CALL EQOSP(RHO02,E02,P02)                                        SUB2 311
      UBARO2=(1.-RHOSTR/RHO02)*(P02/RHOSTR)                            SUB2 312
      IF (UBARO2.GT.0.) GO TO 2669                                     SUB2 313
      WRITE (3,2700) P02,RHO02,E02,R0,Z0                               SUB2 314
      WRITE (3,7002) P1,U1,V1,RHO1,E1,Z1TR1,SBAR1                      SUB2 315
2700  FORMAT (4E16.8)                                                  SUB2 316
2669  CONTINUE                                                         SUB2 317
      UBARO2=SQRT(UBARO2)                                              SUB2 318
      IF (E02.LT.1.E-5) GO TO 273                                      SUB2 319
      IF (ABS((E02-E0)/E02).LT.EPS4) GO TO 273                         SUB2 320
      IF (ABS(DE0).GT..01*EPS4) GO TO 275                              SUB2 321
273   IF (ABS((RHO02-RHO0)/RHO02).LE.EPS3) GO TO 285                   SUB2 322
      IF (ABS(DRHO0).LT.EPS3) GO TO 285                                SUB2 323
275   ITS22=ITS22-1                                                    SUB2 324
      IF (ITS22.GT.0) GO TO 280                                        SUB2 325
      WRITE (3,278) ITS2                                               SUB2 326
278   FORMAT (35HOE AND RHO FAILED TO CONVERGE AFTER,I4,6H TRIES)      SUB2 327
      WRITE (3,279) I,K,Z0,R0,E0,RHO0,P0,E02,RHO02,P02                 SUB2 328
279   FORMAT (1X2HI=,I4,4X2HK=,I4/1X4HZ0 =,E15.8,4X4HR0 =,E15.8,4X4HE0 =SUB2 329
     1,E15.8,4X6HRHO0 =,E15.8,4X4HP0 =,E15.8/1X4HE02=,E15.8,4X6HRHO02=,ESUB2 330
     115.8,4X4HP02=,E15.8/1H1)                                         SUB2 331
      STOP                                                             SUB2 332
280   E0=E02                                                           SUB2 333
      RHO0=RHO02                                                       SUB2 334
      P0=P02                                                           SUB2 335
      UBAR0=UBARO2                                                     SUB2 336
      GO TO 250                                                        SUB2 337
285   E0=E02                                                           SUB2 338
      RHO0=RHO02                                                       SUB2 339
      UBAR0=UBARO2                                                     SUB2 340
      A0=SQRT(PFR+P02*PFE/RHO0**2)                                     SUB2 341
C                                                                      SUB2 342
C                                                                      SUB2 343
      CALL DVCHK(KQ)                                                   SUB2 344
      KICK=285                                                         SUB2 345
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      IF (KQ.EQ.1) GO TO 9980                                          SUB2 346
      IF (K.EQ.2) GO TO 286                                            SUB2 347
      VBARO=VP*LO                                                      SUB2 348
      GO TO 287                                                        SUB2 349
286   VBARO=0.                                                         SUB2 350
287   CONTINUE                                                         SUB2 351
295   PO=PO2                                                           SUB2 352
      IF (K.EQ.2) GO TO 296                                            SUB2 353
      UBARO=UBARS1-UBARO                                               SUB2 354
296   CONTINUE                                                         SUB2 355
      UTO=(RHOO*UBARO-RHOSTR*UBARS1)/(RHOO-RHOSTR)                     SUB2 356
      VO=MO*UBARO+LO*VBARO                                             SUB2 357
      UO=LO*UBARO-MO*VBARO                                             SUB2 358
      UBAR=.5*(UTOH+UTO)                                               SUB2 359
      IF (ABS((UBAR-UTOH)/UBAR).LE.EPS6) GO TO 350                     SUB2 360
      IF (ABS(UBAR-UTOH).LT.EPS6) GO TO 350                            SUB2 361
      ITS44=ITS44-1                                                    SUB2 362
      IF (ITS44.GT.0) GO TO 325                                        SUB2 363
      WRITE (3,297) ITS4,UTOH,UTO                                      SUB2 364
297   FORMAT (30HOUBAR FAILED TO CONVERGE AFTER,I4,6H TRIES/1X5HUTOH=,E1SUB2 365
     15.8,4X4HUTO=,E15.8)                                              SUB2 366
      CALL EXIT                                                        SUB2 367
      WRITE (3,279) I,K,ZO,RO,EO,RHOO,PO,EO2,RHOO2,PO2                 SUB2 368
C                                                                      SUB2 369
C          INIT. FOR MORE U BAR ITERATIONS                             SUB2 370
C                                                                      SUB2 371
325   UTOH=UBAR                                                        SUB2 372
      AVMO=(TAB(I,10,K)+MO)*.5                                         SUB2 373
      AVLO=(TAB(I,9,K)+LO)*.5                                          SUB2 374
      ZO=TAB(I,1,K)+UBAR*H*AVMO-VBARS*AVLO*H                           SUB2 375
      RO=TAB(I,2,K)+UBAR*H*AVLO-VBARS*AVMO*H                           SUB2 376
      LO=AVLO                                                          SUB2 377
      MO=AVMO                                                          SUB2 378
      GO TO 195                                                        SUB2 379
C                                                                      SUB2 380
C          ONE SHOCK POINT HAS CONVERGED                               SUB2 381
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C                                                                       SUB2  382
350   TAB2(I,1,K)=Z0                                                    SUB2  383
      TAB2(I,2,K)=R0                                                    SUB2  384
      TAB2(I,3,K)=P0                                                    SUB2  385
      TAB2(I,4,K)=U0                                                    SUB2  386
      TAB2(I,5,K)=V0                                                    SUB2  387
      TAB2(I,6,K)=RHO0                                                  SUB2  388
      TAB2(I,7,K)=E0                                                    SUB2  389
      TAB2(I,8,K)=A0                                                    SUB2  390
      TAB2(I,9,K)=L0                                                    SUB2  391
      TAB2(I,10,K)=M0                                                   SUB2  392
      TAB2(I,11,K)=UBAR0                                                SUB2  393
      TAB2(I,12,K)=VBAR0                                                SUB2  394
      TAB2(I,13,K)=UT0                                                  SUB2  395
      KICK=500                                                          SUB2  396
      CALL DVCHK(KQ)                                                    SUB2  397
      IF (KQ.EQ.1) GO TO 9980                                           SUB2  398
C                                                                       SUB2  399
C                                                                       SUB2  400
500   CONTINUE                                                          SUB2  401
505   CONTINUE                                                          SUB2  402
      RETURN                                                            SUB2  403
9980  WRITE (3,9985) KICK                                               SUB2  404
9985  FORMAT (32H0DIVIDE CHECK NEAR STATEMENT NO.,I5,15H IN SUBR. SHOCK/SUB2  405
     11H1)                                                              SUB2  406
      CALL EXIT                                                         SUB2  407
      RETURN                                                            SUB2  408
      END                                                               SUB2  409
      SUBROUTINE FGOF1(ZX,RX,SS,QQ)                                     SUB3    1
C                                                                       SUB3    2
C         COMPUTES S1,Q1 FOR SHOCK LINE                                 SUB3    3
C         ITERATION FOR Z1,R1                                           SUB3    4
C                                                                       SUB3    5
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPSSUB3    6
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LENSUB3    7
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS       SUB3    8
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      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2SUB3    9
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(SUB3   10
     115,4),RPART(15,2)                                                SUB3   11
C                                                                       SUB3   12
C                                                                       SUB3   13
      COMMON ZO,RO,PO,UO,VO,LO,MO,RHOO,EO,AO,UBARO,VBARO                SUB3   14
C                                                                       SUB3   15
      COMMON NP,NT,NR,NI,NDEL,ISUB                                      SUB3   16
C                                                                       SUB3   17
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                   SUB3   18
      COMMON DIRCOS                                                     SUB3   19
      COMMON TIME                                                       SUB3   20
      COMMON IRARF                                                      SUB3   21
      COMMON KSTOP                                                      SUB3   22
      COMMON TPSI                                                       SUB3   23
      COMMON KKK                                                        SUB3   24
      REAL LO,MO,LENGTH,MU,KO                                           SUB3   25
C                                                                       SUB3   26
      DIMENSION ANS(6)                                                  SUB3   27
C                                                                       SUB3   28
      REAL LO,MO                                                        SUB3   29
      CALL DBLTRP(ZX,RX,ANS)                                            SUB3   30
      U1=ANS(2)                                                         SUB3   31
      V1=ANS(3)                                                         SUB3   32
      A1=ANS(6)                                                         SUB3   33
      SS=ZX-ZO+H*(V1+A1*DIRCOS)                                         SUB3   34
      QQ=RX-RO+H*(U1+A1*LO)                                             SUB3   35
      RETURN                                                            SUB3   36
      END                                                               SUB3   37
      SUBROUTINE INTER                                                  SUB4    1
C         COMPUTES INTERIOR REGION POINTS                               SUB4    2
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPSSUB4    3
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LENSUB4    4
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS       SUB4    5
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2SUB4    6
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(SUB4    7
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     115,4),RPART(15,2)                                                    SUB4    8
C                                                                          SUB4    9
C                                                                          SUB4   10
      COMMON ZO,RO,PO,UO,VO,LO,MO,RHOO,EO,AO,UBARO,VBARO                   SUB4   11
C                                                                          SUB4   12
      COMMON NP,NT,NR,NI,NDEL,ISUB                                         SUB4   13
C                                                                          SUB4   14
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                      SUB4   15
      COMMON DIRCOS                                                        SUB4   16
      COMMON TIME                                                          SUB4   17
      COMMON IRARF                                                         SUB4   18
      COMMON KSTOP                                                         SUB4   19
      COMMON TPSI                                                          SUB4   20
      COMMON KKK                                                           SUB4   21
C                                                                          SUB4   22
C                                                                          SUB4   23
      REAL LO,MO,LENGTH,MU,KO                                              SUB4   24
      DIMENSION ANS(6),LL(3),ZI(11),RI(11),PI(11),UI(11),VI(11),RHOI(11)SUB4   25
     1,EI(11),AI(11),PUR(11),PVR(11),PAR(11),PUZ(11),PVZ(11),PAZ(11),PSISUB4   26
     1(7),SPSI(11),CPSI(11),S(11)                                          SUB4   27
C                                                                          SUB4   28
      EXTERNAL FGOFI,FGOF5                                                 SUB4   29
      INTEGER CHECK,CHECK2                                                 SUB4   30
C                                                                          SUB4   31
1     FORMAT (1H1)                                                         SUB4   32
      EPS=.0000001                                                         SUB4   33
      DO 905 J=1,20                                                        SUB4   34
      DO 900 I=1,20                                                        SUB4   35
      M=TEST(Z(I),R(J))                                                    SUB4   36
      ZO=Z(I)                                                              SUB4   37
      RO=R(J)                                                              SUB4   38
      KICK=1                                                               SUB4   39
      CALL DVCHK(KQ)                                                       SUB4   40
      IF (KQ.EQ.1) GO TO 9980                                              SUB4   41
      IF (M.EQ.3.AND.ZO.LT.EPS.AND.ABS(RO-RADIUS).LT.EPS)M=1               SUB4   42
      IF (M.NE.1) GO TO 900                                                SUB4   43
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      DO 2 L=1,NP                                                       SUB4   44
      IF (TAB(L,14,1).LT.0.) GO TO 20                                   SUB4   45
2     CONTINUE                                                          SUB4   46
      IF (IRARF.EQ.1) GO TO 20                                          SUB4   47
      DO 3 N=1,NR                                                       SUB4   48
      IF (R(J).GT.RARF(N,2)) GO TO 5                                    SUB4   49
3     CONTINUE                                                          SUB4   50
      GO TO 6                                                           SUB4   51
5     CONTINUE                                                          SUB4   52
      M=PICK(Z(I),R(J),3)                                               SUB4   53
      FF=R(J)-RARF(M+1,2)-(RARF(M,2)-RARF(M+1,2))*(Z(I)-RARF(M+1,1))/(RASUB4   54
     1RF(M,1)-RARF(M+1,1))                                              SUB4   55
      IF (FF.GT.0.) GO TO 20                                            SUB4   56
6     CONTINUE                                                          SUB4   57
      PO2=RARF(1,3)                                                     SUB4   58
      UO2=RARF(1,4)                                                     SUB4   59
      VO2=RARF(1,5)                                                     SUB4   60
      RHOO2=RARF(1,6)                                                   SUB4   61
      EO2=RARF(1,7)                                                     SUB4   62
      AO2=RARF(1,8)                                                     SUB4   63
      GO TO 870                                                         SUB4   64
20    CONTINUE                                                          SUB4   65
      CALL GUESS(2,KOD,Z0,R0,I,J,ZZ,RR,DZ,DR)                          SUB4   66
      IF (KOD.EQ.1) GO TO 580                                           SUB4   67
      WRITE (3,575) I,J,Z0,R0                                           SUB4   68
575   FORMAT (41HONO GUESS FOUND FOR INTERIOR REGION POINT/3HOI=I4,6X2HKSUB4   69
     1=,I4,10X3HZ0=,E15.8,10X3HR0=,E15.8/1H1)                           SUB4   70
      CALL EXIT                                                         SUB4   71
580   CALL DBLTRP(ZZ,RR,ANS)                                            SUB4   72
C                                                                       SUB4   73
C          INITIALIZE FOR 1 POINT                                       SUB4   74
C                                                                       SUB4   75
      PSI(1)=0.                                                         SUB4   76
      PSI(3)=1.0472                                                     SUB4   77
      PSI(4)=2.0944                                                     SUB4   78
      PSI(6)=4.18879                                                    SUB4   79
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      PSI(7)=5.23599                                                    SUB4   80
      NBIC=0                                                            SUB4   81
      ITI22=ITI2                                                        SUB4   82
      PO=ANS(1)                                                         SUB4   83
      UO=ANS(2)                                                         SUB4   84
      VO=ANS(3)                                                         SUB4   85
      RHOO=ANS(4)                                                       SUB4   86
      EO=ANS(5)                                                         SUB4   87
      AO=ANS(6)                                                         SUB4   88
      KICK=580                                                          SUB4   89
      CALL DVCHK(KQ)                                                    SUB4   90
      IF (KQ.EQ.1) GO TO 9980                                           SUB4   91
590   IF (ABS(RO).GT.EPS) GO TO 594                                     SUB4   92
      L1=1                                                              SUB4   93
      LL(1)=4                                                           SUB4   94
      LL(2)=6                                                           SUB4   95
      LLL=2                                                             SUB4   96
      GO TO 620                                                         SUB4   97
C                                                                       SUB4   98
C                                                                       SUB4   99
594   IF (ABS(RO-RADIUS).GT.EPS) GO TO 600                              SUB4  100
      IF (ZO.GT.EPS) GO TO 610                                          SUB4  101
      L1=2                                                              SUB4  102
      LL(1)=3                                                           SUB4  103
      LL(2)=7                                                           SUB4  104
      LLL=2                                                             SUB4  105
      GO TO 620                                                         SUB4  106
600   IF (RO.LE.RADIUS.OR.ABS(ZO).GT.EPS) GO TO 610                     SUB4  107
      L1=3                                                              SUB4  108
      LL(1)=6                                                           SUB4  109
      LL(2)=7                                                           SUB4  110
      LLL=2                                                             SUB4  111
      GO TO 620                                                         SUB4  112
C                                                                       SUB4  113
C                                                                       SUB4  114
610   L1=4                                                              SUB4  115
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      LL(1)=1                                                                  SUB4 116
      LL(2)=4                                                                  SUB4 117
      LL(3)=6                                                                  SUB4 118
      LLL=3                                                                    SUB4 119
C                                                                              SUB4 120
C          ITERATE FOR I VALUES                                                SUB4 121
C                                                                              SUB4 122
619   CONTINUE                                                                 SUB4 123
620   DO 630 KK=1,LLL                                                          SUB4 124
      LUMP=L1                                                                  SUB4 125
      ISUB=LL(KK)                                                              SUB4 126
621   CONTINUE                                                                 SUB4 127
      TPSI=PSI(ISUB)                                                           SUB4 128
      SPSI(ISUB)=SIN(TPSI)                                                     SUB4 129
      CPSI(ISUB)=COS(TPSI)                                                     SUB4 130
      CALL NRIT2(ZI(ISUB),RI(ISUB),ZZ,DZ,RR,DR,EPI1,EPI2,FGOFI,ITI1,KODESUB4 131
     1)                                                                        SUB4 132
      IF (KODE.NE.0) GO TO 6200                                                SUB4 133
625   CALL DBLTRP(ZI(ISUB),RI(ISUB),ANS)                                       SUB4 134
      PI(ISUB)=ANS(1)                                                          SUB4 135
      UI(ISUB)=ANS(2)                                                          SUB4 136
      VI(ISUB)=ANS(3)                                                          SUB4 137
      RHOI(ISUB)=ANS(4)                                                        SUB4 138
      EI(ISUB)=ANS(5)                                                          SUB4 139
      AI(ISUB)=ANS(6)                                                          SUB4 140
630   CONTINUE                                                                 SUB4 141
C                                                                              SUB4 142
C                                                                              SUB4 143
      KICK=630                                                                 SUB4 144
      CALL DVCHK(KQ)                                                           SUB4 145
      IF (KQ.EQ.1) GO TO 9980                                                  SUB4 146
      GO TO 6400                                                               SUB4 147
C                                                                              SUB4 148
C     BICHARACTERISTIC SELECTION SCHEME                                        SUB4 149
C                                                                              SUB4 150
7000  WRITE (3,7001)                                                           SUB4 151
```

```
7001  FORMAT (41HOBICHARACTERISTIC SELECTION SCHEME FAILED)                SUB4 152
      WRITE (3,614) ZO,RO                                                  SUB4 153
      WRITE (3,7002) (PSI(MNMN),MNMN=1,7),UA,VA,AA,UB,VB,AB,UC,VC,AC,ZZ,SUB4 154
     1DZ,RR,DR,ANG1,ANG2                                                   SUB4 155
      SUB=ISUB                                                             SUB4 156
      WRITE (3,7002) SUB,ZI(ISUB),RI(ISUB)                                 SUB4 157
7002  FORMAT (4E16.8)                                                      SUB4 158
      CALL EXIT                                                            SUB4 159
6200  CONTINUE                                                             SUB4 160
      IF (NBIC.NE.0) GO TO 7000                                            SUB4 161
      IF (L1.NE.2.OR.LL(1).EQ.1) GO TO 7300                                SUB4 162
      LL(1)=1                                                              SUB4 163
      GO TO 619                                                            SUB4 164
7300  CONTINUE                                                             SUB4 165
      IF (L1.NE.3) GO TO 7310                                              SUB4 166
      IF (PSI(6).GT.4.2) GO TO 7310                                        SUB4 167
      PSI(6)=5.75959                                                       SUB4 168
      GO TO 619                                                            SUB4 169
7310  CONTINUE                                                             SUB4 170
      CALL DBLTRP(ZZ,RR,ANS)                                               SUB4 171
      UA=ANS(2)                                                            SUB4 172
      VA=ANS(3)                                                            SUB4 173
      AA=ANS(6)                                                            SUB4 174
      ZZZ=ZZ+DZ                                                            SUB4 175
      CALL DBLTRP(ZZZ,RR,ANS)                                              SUB4 176
      UB=ANS(2)                                                            SUB4 177
      VB=ANS(3)                                                            SUB4 178
      AB=ANS(6)                                                            SUB4 179
      RRR=RR+DR                                                            SUB4 180
      CALL DBLTRP(ZZ,RRR,ANS)                                              SUB4 181
      UC=ANS(2)                                                            SUB4 182
      VC=ANS(3)                                                            SUB4 183
      AC=ANS(6)                                                            SUB4 184
      MM=0                                                                 SUB4 185
      TPSI=PSI(ISUB)                                                       SUB4 186
      XB=ZZ                                                                SUB4 187
```

```
      YB=RR                                                              SUB4 188
      NOM=5                                                              SUB4 189
      CA=NOM                                                             SUB4 190
      DO 6210 LM=1,2                                                     SUB4 191
6201  DTPSI=.01745                                                       SUB4 192
      DTPSI=CA*DTPSI                                                     SUB4 193
      TPSI=TPSI+DTPSI                                                    SUB4 194
      A1=1.+H*(VB-VA+(AB-AA)*SIN(TPSI))/DZ                               SUB4 195
      B1=H*(VC-VA+(AC-AA)*SIN(TPSI))/DR                                  SUB4 196
      C1=-(ZZ-ZO+H*(VA+AA*SIN(TPSI)))                                    SUB4 197
      A2=H*(UB-UA+(AB-AA)*COS(TPSI))/DZ                                  SUB4 198
      B2=1.+H*(UC-UA+(AC-AA)*COS(TPSI))/DR                               SUB4 199
      C2=-(RR-RO+H*(UA+AA*COS(TPSI)))                                    SUB4 200
      DET=A1*B2-A2*B1                                                    SUB4 201
      DELX=(B2*C1-B1*C2)/DET                                             SUB4 202
      DELY=(A1*C2-A2*C1)/DET                                             SUB4 203
C                                                                        SUB4 204
C          TEST FOR SAME REGION                                          SUB4 205
C                                                                        SUB4 206
      XB1=XB+DELX                                                        SUB4 207
      YB1=YB+DELY                                                        SUB4 208
      M=COMP(XB,YB,XB1,YB1)                                              SUB4 209
      IF (LM.EQ.2) GO TO 6700                                            SUB4 210
      IF (M.EQ.1) GO TO 6203                                             SUB4 211
      GO TO 6800                                                         SUB4 212
6700  CONTINUE                                                           SUB4 213
      IF (M.NE.1) GO TO 6203                                             SUB4 214
6800  CONTINUE                                                           SUB4 215
      MM=MM+1                                                            SUB4 216
      IF (MM.LE.360/NOM) GO TO 6201                                      SUB4 217
612   WRITE (3,613) ITI1                                                 SUB4 218
613   FORMAT (44HOFAILED TO FIND 2 POINTS IN THE SAME REGION 21H IN SUBRSUB4 219
     1. NRIT2 AFTER,I4,6H TRIES)                                         SUB4 220
      WRITE (3,614) ZO,RO                                                SUB4 221
614   FORMAT (1X5HZO  =,E15.8,4X5HRO  =,E15.8)                           SUB4 222
      WRITE (3,6144) LM                                                  SUB4 223
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```
      WRITE (3,6145) M                                                  SUB4 224
      WRITE (3,6146) KODE                                               SUB4 225
      WRITE (3,7002) XB,YB,XB1,YB1                                      SUB4 226
6144  FORMAT (4H LM=,I4)                                                SUB4 227
6145  FORMAT (3H M=,I4)                                                 SUB4 228
6146  FORMAT (6H KODE=,I4)                                              SUB4 229
      CALL EXIT                                                         SUB4 230
6203  GO TO (6204,6205),LM                                              SUB4 231
6204  ANG1=TPSI                                                         SUB4 232
      MM=0                                                              SUB4 233
      GO TO 6210                                                        SUB4 234
6205  ANG2=TPSI-DTPSI                                                   SUB4 235
6210  CONTINUE                                                          SUB4 236
      AL=LLL+1                                                          SUB4 237
      DO 6300 KK=1,LLL                                                  SUB4 238
      ISUB=LL(KK)                                                       SUB4 239
      AK=KK                                                             SUB4 240
      PSI(ISUB)=ANG1+(ANG2-ANG1)*AK/AL                                  SUB4 241
6300  CONTINUE                                                          SUB4 242
      NBIC=1                                                            SUB4 243
      GO TO 619                                                         SUB4 244
C                                                                       SUB4 245
C                                                                       SUB4 246
6400  CONTINUE                                                          SUB4 247
      IF (L1.EQ.2.OR.L1.EQ.3) GO TO 642                                 SUB4 248
      CALL NRIT2(ZI(8),RI(8),ZZ,DZ,RR,DR,EPI1,EPI2,FGOF5,ITI1,KODE)     SUB4 249
      IF (KODE.EQ.0) GO TO 635                                          SUB4 250
      ISUB=8                                                            SUB4 251
      WRITE (3,622) ITI1,I,J,ISUB,ZO,RO,ZZ,RR,ZI(8),RI(8)               SUB4 252
622   FORMAT (27HOFAILED TO FIND ZI,RI AFTER,I4,6H TRIES,3X2HI=,I4,3X2HJSUB4 253
     1=,I4,3X5HISUB=,I4/1X3HZO=,E15.8,6X3HRO=,E15.8/1X3HZZ=,E15.8,6X3HRRSUB4 254
     1=,E15.8/1X3HZI=,E15.8,6X3HRI=,E15.8/1H1)                          SUB4 255
      CALL EXIT                                                         SUB4 256
C                                                                       SUB4 257
C                                                                       SUB4 258
635   CALL OBLTRP(ZI(8),RI(8),ANS)                                      SUB4 259
```

```
      PI(8)=ANS(1)                                                      SUB4 260
      UI(8)=ANS(2)                                                      SUB4 261
      VI(8)=ANS(3)                                                      SUB4 262
      RHOI(8)=ANS(4)                                                    SUB4 263
      EI(8)=ANS(5)                                                      SUB4 264
      AI(8)=ANS(6)                                                      SUB4 265
C                                                                       SUB4 266
642   DO 670 IL=1,LLL                                                   SUB4 267
      NN=LL(IL)                                                         SUB4 268
      M=PART(1,ZI(NN),RI(NN),ZX,RX,DELTA,NDEL)                          SUB4 269
      IF (M.EQ.1) GO TO 645                                             SUB4 270
      PUR(NN)=0.                                                        SUB4 271
      PVR(NN)=0.                                                        SUB4 272
      GO TO 648                                                         SUB4 273
645   CALL DBLTRP(ZX,RX,ANS)                                            SUB4 274
      DEN=RX-RI(NN)                                                     SUB4 275
      PUR(NN)=(ANS(2)-UI(NN))/DEN                                       SUB4 276
      PVR(NN)=(ANS(3)-VI(NN))/DEN                                       SUB4 277
648   M=PART(2,ZI(NN),RI(NN),ZX,RX,DELTA,NDEL)                          SUB4 278
      IF (M.EQ.1) GO TO 650                                             SUB4 279
      PUZ(NN)=0.                                                        SUB4 280
      PVZ(NN)=0.                                                        SUB4 281
      GO TO 655                                                         SUB4 282
650   CALL DBLTRP(ZX,RX,ANS)                                            SUB4 283
      DEN=ZX-ZI(NN)                                                     SUB4 284
      PUZ(NN)=(ANS(2)-UI(NN))/DEN                                       SUB4 285
      PVZ(NN)=(ANS(3)-VI(NN))/DEN                                       SUB4 286
655   S(NN)=SPSI(NN)**2*PUR(NN)-CPSI(NN)*SPSI(NN)*(PVR(NN)+PUZ(NN))+CPSI SUB4 287
     1(NN)**2*PVZ(NN)                                                   SUB4 288
C                                                                       SUB4 289
      IF (ABS(RI(NN)).GT.EPS) GO TO 660                                 SUB4 290
      CON=PUR(NN)                                                       SUB4 291
      GO TO 662                                                         SUB4 292
660   CON=UI(NN)/RI(NN)                                                 SUB4 293
662   S(NN)=-RHOI(NN)*H*AI(NN)**2*(S(NN)+CON)+PI(NN)+RHOI(NN)*AI(NN)*CPSSUB4 294
     1I(NN)*UI(NN)+RHOI(NN)*AI(NN)*SPSI(NN)*VI(NN)                      SUB4 295
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670   CONTINUE                                                                SUB4 296
1005  CONTINUE                                                                SUB4 297
1002  FORMAT (4E16.8)                                                         SUB4 298
      KICK=670                                                                SUB4 299
      CALL DVCHK(KQ)                                                          SUB4 300
      IF (KQ.EQ.1) GO TO 9980                                                 SUB4 301
C                                                                             SUB4 302
C         COMPUTE NEW P,U,V                                                   SUB4 303
C                                                                             SUB4 304
      GO TO (690,692,695,698),L1                                              SUB4 305
690   DU0=0.                                                                  SUB4 306
      U02=0.                                                                  SUB4 307
      V02=(S(4)-S(6))/(RHOI(4)*AI(4)*SPSI(4)-RHOI(6)*AI(6)*SPSI(6))           SUB4 308
      P02=S(4)-RHOI(4)*AI(4)*SPSI(4)*V02                                      SUB4 309
      GO TO 700                                                               SUB4 310
C                                                                             SUB4 311
C                                                                             SUB4 312
692   DP0=0.                                                                  SUB4 313
      P02=0.                                                                  SUB4 314
      L=LL(1)                                                                 SUB4 315
      TMP1=RHOI(7)*RHOI(L)*AI(7)*AI(L)*(CPSI(7)*SPSI(L)-CPSI(L)*SPSI(7))      SUB4 316
      V02=(S(L)*RHOI(7)*AI(7)*CPSI(7)-S(7)*RHOI(L)*AI(L)*CPSI(L))/TMP1        SUB4 317
      U02=(S(L)-RHOI(L)*AI(L)*SPSI(L)*V02)/(RHOI(L)*AI(L)*CPSI(L))            SUB4 318
      GO TO 700                                                               SUB4 319
C                                                                             SUB4 320
C                                                                             SUB4 321
695   DP0=0.                                                                  SUB4 322
      P02=0.                                                                  SUB4 323
      TMP1=RHOI(7)*RHOI(6)*AI(7)*AI(6)*(CPSI(7)*SPSI(6)-SPSI(7)*CPSI(6))      SUB4 324
      V02=(S(6)*RHOI(7)*AI(7)*CPSI(7)-S(7)*RHOI(6)*AI(6)*CPSI(6))/TMP1        SUB4 325
      U02=(S(6)-RHOI(6)*AI(6)*SPSI(6)*V02)/(RHOI(6)*AI(6)*CPSI(6))            SUB4 326
      GO TO 700                                                               SUB4 327
698   CONTINUE                                                                SUB4 328
      L=LL(1)                                                                 SUB4 329
      TMP1=RHOI(4)*AI(4)*CPSI(4)-RHOI(L)*AI(L)*CPSI(L)                        SUB4 330
      TMP2=RHOI(4)*AI(4)*SPSI(4)-RHOI(L)*AI(L)*SPSI(L)                        SUB4 331
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      TMP3=RHOI(6)*AI(6)*CPSI(6)-RHOI(L)*AI(L)*CPSI(L)                    SUB4 332
      TMP4=RHOI(6)*AI(6)*SPSI(6)-RHOI(L)*AI(L)*SPSI(L)                    SUB4 333
      V02=((S(4)-S(L))*TMP3-(S(6)-S(L))*TMP1)/(TMP3*TMP2-TMP1*TMP4)       SUB4 334
      U02=(S(4)-S(L)-TMP2*V02)/TMP1                                       SUB4 335
      P02=S(6)-RHOI(6)*AI(6)*CPSI(6)*U02-RHOI(6)*AI(6)*SPSI(6)*V02        SUB4 336
700   KICK=700                                                            SUB4 337
      CALL DVCHK(KQ)                                                      SUB4 338
      IF (KQ.EQ.1) GO TO 9980                                             SUB4 339
C                                                                         SUB4 340
C                                                                         SUB4 341
C         ITERATE FOR RHOO,EO                                             SUB4 342
C                                                                         SUB4 343
705   ITI33=ITI3                                                         SUB4 344
      ITI44=ITI4                                                          SUB4 345
      KM=1                                                                SUB4 346
708   CONTINUE                                                            SUB4 347
      CALL EQOSI(P02,PGRHO,PGE,BIGG,CHECK,KRTT,A02,E02,RHO02,KM,EPS)      SUB4 348
      IF (KRTT.EQ.1) GO TO 871                                            SUB4 349
725   T1=RHOO-RHOI(8)                                                     SUB4 350
      T2=PI(8)/RHOI(8)**2                                                 SUB4 351
      BIGH=EI(8)+T2*T1-EO                                                 SUB4 352
      PHE=-1.                                                             SUB4 353
      PHRHO=T2                                                            SUB4 354
      KICK=725                                                            SUB4 355
      CALL DVCHK(KQ)                                                      SUB4 356
      IF (KQ.EQ.1) GO TO 9980                                             SUB4 357
C                                                                         SUB4 358
C         COMPUTE NEW EO,RHOO                                             SUB4 359
C                                                                         SUB4 360
      DOWN=PGE*PHRHO-PGRHO*PHE                                            SUB4 361
      DEO=(-BIGG*PHRHO+BIGH*PGRHO)/DOWN                                   SUB4 362
      DRHOO=(-BIGH*PGE+BIGG*PHE)/DOWN                                     SUB4 363
      E02=EO+DEO                                                          SUB4 364
      RHO02=RHOO+DRHOO                                                    SUB4 365
C                                                                         SUB4 366
C         CHECK E02,RHO02 FOR CONVERGENCE                                 SUB4 367
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      KICK=726                                                          SUB4 368
      CALL DVCHK(KQ)                                                    SUB4 369
      IF (KQ.EQ.1) GO TO 9980                                           SUB4 370
C                                                                       SUB4 371
      IF (ABS(DEO/EO).LT.EPI7) GO TO 726                                SUB4 372
      IF (ABS(DEO).GT..01*EPI7) GO TO 730                               SUB4 373
726   IF (ABS(DRHOO/RHOO).LE.EPI6) GO TO 740                            SUB4 374
      IF (ABS(DRHOO).LT.EPI6) GO TO 740                                 SUB4 375
730   ITI33=ITI33-1                                                     SUB4 376
      IF (ITI33.NE.0) GO TO 735                                         SUB4 377
      WRITE (6,732) ITI3,I,J,ZO,RO,PO,UO,VO,RHOO,EO,PO2,UO2,VO2,RHOO2,EOSUB4 378
     12                                                                 SUB4 379
732   FORMAT (33HOEO,RHOO FAILED TO CONVERGE AFTER,I4,6H TRIES/1X2HI=,I4SUB4 380
     1,4X2HJ=,I4,4X2HZ=,E15.8,4X2HR=,E15.8/5X2HPO18X2HUO18X2HVO18X4HRHOOSUB4 381
     116X2HEO/5X3HPO217X3HUO217X3HVO217X5HRHOO215X3HEO2//(5E20.8))     SUB4 382
      WRITE (3,1)                                                       SUB4 383
      CALL EXIT                                                         SUB4 384
735   EO=EO2                                                            SUB4 385
      RHOO=RHOO2                                                        SUB4 386
      KM=0                                                              SUB4 387
      GO TO 708                                                         SUB4 388
C                                                                       SUB4 389
C         CHECK FOR PROPER EQUATIONS                                    SUB4 390
C                                                                       SUB4 391
740   CONTINUE                                                          SUB4 392
      PGE=-PGE                                                          SUB4 393
      PGRHO=-PGRHO                                                      SUB4 394
      IF (RHOO2.GE.RHOSTR) GO TO 750                                    SUB4 395
      IF (EO2.LT.EPRS) GO TO 750                                        SUB4 396
742   CHECK2=0                                                          SUB4 397
      GO TO 752                                                         SUB4 398
750   CHECK2=1                                                          SUB4 399
752   AO2=SQRT(+PGRHO+PO2*PGE/RHOO2**2)                                 SUB4 400
      IF (CHECK.EQ.CHECK2) GO TO 870                                    SUB4 401
      ITI44=ITI44-1                                                     SUB4 402
      IF (ITI44.NE.0) GO TO 770                                         SUB4 403
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      WRITE (3,755) ITI4,I,J,ZO,RO,PO,UO,VO,RHOO,EO,AO,PO2,UO2,VO2,RHOO2SUB4  404
     1,EO2,AO2                                                          SUB4  405
755   FORMAT (38HOFAILED TO USE CORRECT EQUATIONS AFTER,I4,6H TRIES/1X2HSUB4  406
     1I=,I4,4X2HJ=,I4,4X2HZ=,E15.8,4X2HR=,E15.8/5X2HPO18X2HUO18X2HVO18X4SUB4  407
     1HRHOO16X2HEO18X2HAO/5X3HPO217X3HUO217X3HVO217X5HRHOO215X3HEO217X3HSUB4  408
     1AO2//(6E20.8))                                                    SUB4  409
      WRITE (3,1)                                                       SUB4  410
      CALL EXIT                                                         SUB4  411
770   KICK=770                                                          SUB4  412
      CALL DVCHK(KQ)                                                    SUB4  413
      IF (KQ.EQ.1) GO TO 9980                                           SUB4  414
      ITI33=ITI3                                                        SUB4  415
      GO TO 735                                                         SUB4  416
C                                                                       SUB4  417
C          ALL VALUES HAVE CONVERGED FOR 1 INTERIOR POINT               SUB4  418
C                                                                       SUB4  419
871   KRTT=0                                                            SUB4  420
870   XMESH2(I,J,1)=PO2                                                 SUB4  421
      XMESH2(I,J,2)=UO2                                                 SUB4  422
      XMESH2(I,J,3)=VO2                                                 SUB4  423
      XMESH2(I,J,4)=RHOO2                                               SUB4  424
      XMESH2(I,J,5)=EO2                                                 SUB4  425
      XMESH2(I,J,6)=AO2                                                 SUB4  426
      KICK=900                                                          SUB4  427
      CALL DVCHK(KQ)                                                    SUB4  428
      IF (KQ.EQ.1) GO TO 9980                                           SUB4  429
900   CONTINUE                                                          SUB4  430
905   CONTINUE                                                          SUB4  431
      IF (ITS3.EQ.1) GO TO 950                                          SUB4  432
      CALL ITRP                                                         SUB4  433
      RETURN                                                            SUB4  434
950   CONTINUE                                                          SUB4  435
9980  WRITE (3,9985) KICK                                               SUB4  436
      WRITE (3,614) ZO,RO                                               SUB4  437
9985  FORMAT (32HODIVIDE CHECK NEAR STATEMENT NO.,I5,15H IN SUBR. INTER/SUB4  438
     11H1)                                                              SUB4  439
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      CALL EXIT                                                          SUB4  440
      RETURN                                                             SUB4  441
      END                                                                SUB4  442
      SUBROUTINE EQOS1(PRHO,PPP,PVV,PEE,TEE,TRHO,KICK)                   SUB5    1
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPSSUB5    2
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LENSUB5    3
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS        SUB5    4
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2SUB5    5
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(SUB5    6
     115,4),RPART(15,2)                                                  SUB5    7
C                                                                        SUB5    8
C                                                                        SUB5    9
      COMMON ZO,RO,PO,UO,VO,LO,MO,RHOO,EO,AO,UBARO,VBARO                 SUB5   10
C                                                                        SUB5   11
      COMMON NP,NT,NR,NI,NDEL,ISUB                                       SUB5   12
C                                                                        SUB5   13
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                    SUB5   14
      COMMON DIRCOS                                                      SUB5   15
      COMMON TIME                                                        SUB5   16
      COMMON IRARF                                                       SUB5   17
      COMMON KSTOP                                                       SUB5   18
      COMMON TPSI                                                        SUB5   19
      COMMON KKK                                                         SUB5   20
C                                                                        SUB5   21
C                                                                        SUB5   22
      REAL LO,MO,LENGTH,MU,KO                                            SUB5   23
      RHO=RHOSTR                                                         SUB5   24
      E=VP**2/8.                                                         SUB5   25
      DO 100 M=1,100                                                     SUB5   26
      ETA=RHO/RHOSTR                                                     SUB5   27
      MU=ETA-1.                                                          SUB5   28
      G=-RHOSTR*(VP/2.)**2+((APR+BPR/(E/(ESTAR*ETA**2)+1.))*E*RHO+BIGAPRSUB5   29
     1*MU+BIGBPR*MU**2)*(1.-RHOSTR/RHO)                                  SUB5   30
      DERIVG=((APR+BPR/(E/(ESTAR*ETA**2)+1.))*E+BIGAPR/RHOSTR+2.*BIGBPR*SUB5   31
     1MU/RHOSTR+2.*E**2*BPR/(ESTAR*ETA**2*(E/(ESTAR*ETA**2)+1.)**2))*(1.SUB5   32
     1-RHOSTR/RHO)+((APR+BPR/(E/(ESTAR*ETA**2)+1.))*E*RHO+BIGAPR*MU+BIGBSUB5   33
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     1PR*MU**2)*RHOSTR/RHO**2                                                  SUB5   34
      DLTRHO=-G/DERIVG                                                         SUB5   35
      RHO=RHO+DLTRHO                                                           SUB5   36
      IF(ABS(DLTRHO).LT.1.E-07) GO TO 101                                      SUB5   37
C     IF (ABS(DLTRHO).LT.1.E-06) GO TO 101                                     SUB5   38
100   CONTINUE                                                                 SUB5   39
      KICK=2200                                                                SUB5   40
      GO TO 9980                                                               SUB5   41
101   CONTINUE                                                                 SUB5   42
      PRHO=RHO                                                                 SUB5   43
      TRHO=RHO                                                                 SUB5   44
      PEE=E                                                                    SUB5   45
      TEE=E                                                                    SUB5   46
      PVV=VP/2.                                                                SUB5   47
      PPP=(APR+BPR/(E/(ESTAR*ETA**2)+1.))*E*RHO+BIGAPR*MU+BIGBPR*MU**2         SUB5   48
9980  RETURN                                                                   SUB5   49
      END                                                                      SUB5   50
      SUBROUTINE EQOS2(PPP,PRHO,PEE)                                           SUB6    1
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPSSUB6    2
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LENSUB6    3
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS          SUB6    4
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2SUB6    5
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(SUB6    6
     115,4),RPART(15,2)                                                        SUB6    7
C                                                                             SUB6    8
C                                                                             SUB6    9
      COMMON ZO,RO,PO,UO,VO,LO,MO,RHOO,EO,AO,UBARO,VBARO                      SUB6   10
C                                                                             SUB6   11
      COMMON NP,NT,NR,NI,NDEL,ISUB                                            SUB6   12
C                                                                             SUB6   13
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                         SUB6   14
      COMMON DIRCOS                                                           SUB6   15
      COMMON TIME                                                             SUB6   16
      COMMON IRARF                                                            SUB6   17
      COMMON KSTOP                                                            SUB6   18
C                                                                             SUB6   19
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      COMMON TPSI                                                       SUB6  20
      COMMON KKK                                                        SUB6  21
C                                                                       SUB6  22
      REAL LO,MO,LENGTH,MU,KO                                           SUB6  23
      P=PPP                                                             SUB6  24
      RHO=PRHO                                                          SUB6  25
      E=PEE                                                             SUB6  26
      ETA=RHO/RHOSTR                                                    SUB6  27
      MU=ETA-1.                                                         SUB6  28
      EE=E/(ESTAR*ETA**2)+1.                                            SUB6  29
      PGRHO=E*(APR+BPR/EE)+BIGAPR/RHOSTR+(2.*BIGBPR*MU)/RHOSTR+(2.*E**2*SUB6  30
     1BPR)/(ESTAR*ETA**2*EE**2)                                         SUB6  31
      PGE=(APR+BPR/EE)*RHO-(E*BPR*RHO)/(ESTAR*ETA**2*EE**2)             SUB6  32
      AR=SQRT(PGRHO+PGE*P/RHO**2)                                       SUB6  33
      RETURN                                                            SUB6  34
      END                                                               SUB6  35
      SUBROUTINE EQOS3(RHO,AA,E,P)                                      SUB7   1
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPSSUB7   2
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LENSUB7   3
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS       SUB7   4
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2SUB7   5
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(SUB7   6
     115,4),RPART(15,2)                                                 SUB7   7
C                                                                       SUB7   8
C                                                                       SUB7   9
      COMMON ZO,RO,PO,UO,VO,LO,MO,RHOO,EO,AO,UBARO,VBARO                SUB7  10
C                                                                       SUB7  11
      COMMON NP,NT,NR,NI,NDEL,ISUB                                      SUB7  12
C                                                                       SUB7  13
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                   SUB7  14
      COMMON DIRCOS                                                     SUB7  15
      COMMON TIME                                                       SUB7  16
      COMMON IRARF                                                      SUB7  17
      COMMON KSTOP                                                      SUB7  18
      COMMON TPSI                                                       SUB7  19
      COMMON KKK                                                        SUB7  20
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C                                                                       SUB7   21
C                                                                       SUB7   22
      REAL LO,MO,LENGTH,MU,KO                                           SUB7   23
70    ETA=RHO/RHOSTR                                                    SUB7   24
      MU=ETA-1.                                                         SUB7   25
      EE=E/(ESTAR*ETA**2)+1.                                            SUB7   26
      IF (RHO.GT.RHOSTR) GO TO 72                                       SUB7   27
      IF (E.GE.EPRS) GO TO 74                                           SUB7   28
72    PGRHO=E*(APR+BPR/EE)+BIGAPR/RHOSTR+(2.*BIGBPR*MU)/RHOSTR+(2.*E**2*SUB7   29
     1BPR)/(ESTAR*ETA**2*EE**2)                                         SUB7   30
      PGE=(APR+BPR/EE)*RHO-(E*BPR*RHO)/(ESTAR*ETA**2*EE**2)             SUB7   31
      GO TO 75                                                          SUB7   32
74    C1=RHOSTR/RHO-1.                                                  SUB7   33
      C2=EXP(-BETA*C1)                                                  SUB7   34
      C3=EXP(-ALPHA*C1**2)                                              SUB7   35
      T1=(BPR*E*RHO)/EE+BIGAPR*MU*C2                                    SUB7   36
      T2=2.*ALPHA*C1*(RHOSTR/(RHO**2))                                  SUB7   37
      T3=BPR*E/EE                                                       SUB7   38
      T4=(2.*E)/(ESTAR*ETA**2*EE)                                       SUB7   39
      T4=T3*T4                                                          SUB7   40
      T5=(BIGAPR*C2)/RHOSTR                                             SUB7   41
      T6=(BIGAPR*MU*BETA*RHOSTR*C2)/(RHO**2)                            SUB7   42
      PGRHO=APR*E+C3*(T1*T2+T3+T4+T5+T6)                                SUB7   43
      T7=(BPR*RHO)/EE                                                   SUB7   44
      PGE=APR*RHO+C3*(T7-T7*(E/(ESTAR*ETA**2*EE)))                      SUB7   45
75    AA=SQRT(PGRHO+PGE*P/RHO**2)                                       SUB7   46
      RETURN                                                            SUB7   47
      END                                                               SUB7   48
      SUBROUTINE EQOSS(PFR,PFE)                                         SUB8    1
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPSSUB8    2
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LENSUB8    3
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS       SUB8    4
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2SUB8    5
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(SUB8    6
     115,4),RPART(15,2)                                                 SUB8    7
C                                                                       SUB8    8
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C                                                                       SUB8    9
      COMMON Z0,R0,P0,U0,V0,L0,M0,RHO0,E0,A0,UBAR0,VBAR0                 SUB8   10
C                                                                       SUB8   11
      COMMON NP,NT,NR,NI,NDEL,ISUB                                      SUB8   12
C                                                                       SUB8   13
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                   SUB8   14
      COMMON DIRCOS                                                     SUB8   15
      COMMON TIME                                                       SUB8   16
      COMMON IRARF                                                      SUB8   17
      COMMON KSTOP                                                      SUB8   18
      COMMON TPSI                                                       SUB8   19
      COMMON KKK                                                        SUB8   20
C                                                                       SUB8   21
C                                                                       SUB8   22
      REAL L0,M0,LENGTH,MU,K0                                           SUB8   23
250   ETA=RHO0/RHOSTR                                                   SUB8   24
      MU=ETA-1.                                                         SUB8   25
C                                                                       SUB8   26
C                                                                       SUB8   27
      EPP=E0/(ESTAR*ETA**2)+1.                                          SUB8   28
      TMP=APR+BPR/EPP                                                   SUB8   29
      TMP1=BPR/(ESTAR*ETA**2*EPP**2)                                    SUB8   30
      PFR=TMP*E0+(BIGAPR+2.*BIGBPR*MU)/RHOSTR+2.*E0**2*TMP1             SUB8   31
      PFE=TMP*RHO0-E0*RHO0*TMP1                                         SUB8   32
      RETURN                                                            SUB8   33
      END                                                               SUB8   34
      SUBROUTINE EQOSP(RHO02,E02,P02)                                   SUB9    1
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPSSUB9    2
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LENSUB9    3
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS       SUB9    4
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2SUB9    5
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(SUB9    6
     115,4),RPART(15,2)                                                 SUB9    7
C                                                                       SUB9    8
C                                                                       SUB9    9
      COMMON Z0,R0,P0,U0,V0,L0,M0,RHO0,E0,A0,UBAR0,VBAR0                SUB9   10
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C                                                                       SUB9   11
      COMMON NP,NT,NR,NI,NDEL,ISUB                                      SUB9   12
C                                                                       SUB9   13
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                   SUB9   14
      COMMON DIRCOS                                                     SUB9   15
      COMMON TIME                                                       SUB9   16
      COMMON IRARF                                                      SUB9   17
      COMMON KSTOP                                                      SUB9   18
      COMMON TPSI                                                       SUB9   19
      COMMON KKK                                                        SUB9   20
C                                                                       SUB9   21
C                                                                       SUB9   22
      REAL LO,MO,LENGTH,MU,KO                                           SUB9   23
      ETA=RHO02/RHOSTR                                                  SUB9   24
      MU=ETA-1.                                                         SUB9   25
      PO2=EO2*RHO02*(APR+(BPR*ESTAR*ETA**2)/(EO2+ESTAR*ETA**2))+BIGAPR*MSUB9   26
     1U+BIGBPR*MU**2                                                    SUB9   27
      RETURN                                                            SUB9   28
      END                                                               SUB9   29
      SUBROUTINE EQOSI(PO2,PGRHO,PGE,BIGG,CHECK,KRTT,AO2,EO2,RHO02,KM,EPSU10    1
     1S)                                                                SU10   2
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPSU10    3
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LENU10    4
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS       SU10   5
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2U10    6
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(U10   7
     115,4),RPART(15,2)                                                 SU10   8
C                                                                       SU10   9
C                                                                       SU10   10
      COMMON ZO,RO,PO,UO,VO,LO,MO,RHOO,EO,AO,UBARO,VBARO                SU10   11
C                                                                       SU10   12
      COMMON NP,NT,NR,NI,NDEL,ISUB                                      SU10   13
C                                                                       SU10   14
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                   SU10   15
      COMMON DIRCOS                                                     SU10   16
      COMMON TIME                                                       SU10   17
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      COMMON IRARF                                                      SU10  18
      COMMON KSTOP                                                      SU10  19
      COMMON TPSI                                                       SU10  20
      COMMON KKK                                                        SU10  21
C                                                                       SU10  22
C                                                                       SU10  23
      REAL LO,MO,LENGTH,MU,KO                                           SU10  24
      INTEGER CHECK,CHECK2                                              SU10  25
      IF (KM.EQ.0) GO TO 708                                            SU10  26
      RHOO=RHOSTR                                                       SU10  27
      TMP1=(APR+BPR)*RHOSTR-PO2/ESTAR                                   SU10  28
      TMP2=SQRT(TMP1**2+4.*PO2*APR*RHOSTR/ESTAR)                        SU10  29
      EO=(-TMP1+TMP2)/(2.*APR*RHOSTR/ESTAR)                             SU10  30
      IF (PO2.GT.EPS) GO TO 708                                         SU10  31
      PO2=0.                                                            SU10  32
      EO2=0.                                                            SU10  33
      RHOO2=RHOSTR                                                      SU10  34
      AO2=SQRT(BIGAPR/RHOSTR)                                           SU10  35
      KRTT=1                                                            SU10  36
      GO TO 870                                                         SU10  37
708   KRTT=0                                                            SU10  38
      ETA=RHOO/RHOSTR                                                   SU10  39
      MU=ETA-1.                                                         SU10  40
      EE=EO/(ESTAR*ETA**2)+1.                                           SU10  41
      IF (RHOO.GT.RHOSTR) GO TO 720                                     SU10  42
      IF (EO.LT.EPRS) GO TO 720                                         SU10  43
715   C1=RHOSTR/RHOO-1.                                                 SU10  44
      BEC1=BETA*C1                                                      SU10  45
      IF (BEC1.LT.10.E10) GO TO 716                                     SU10  46
      C2=0.0                                                            SU10  47
      GO TO 717                                                         SU10  48
716   C2=EXP(-BEC1)                                                     SU10  49
717   C3AL=ALPHA*C1**2                                                  SU10  50
      IF (C3AL.LT.10.E12) GO TO 718                                     SU10  51
      C3=0.0                                                            SU10  52
      GO TO 719                                                         SU10  53
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718   C3=EXP(-C3AL)                                                          SU10  54
719   CONTINUE                                                               SU10  55
      T1=(BPR*E0*RHO0)/EE+BIGAPR*MU*C2                                       SU10  56
      T2=2.*ALPHA*C1*(RHOSTR/(RHO0**2))                                      SU10  57
      T3=BPR*E0/EE                                                           SU10  58
      T4=(2.*E0)/(ESTAR*ETA**2*EE)                                           SU10  59
      T4=T3*T4                                                               SU10  60
      T5=(BIGAPR*C2)/RHOSTR                                                  SU10  61
      T6=(BIGAPR*MU*BETA*RHOSTR*C2)/(RHO0**2)                                SU10  62
      T7=(BPR*RHO0)/EE                                                       SU10  63
      PGRHO=APR*E0+C3*(T1*T2+T3+T4+T5+T6)                                    SU10  64
      PGE=APR*RHO0+C3*(T7-T7*(E0/(ESTAR*ETA**2*EE)))                         SU10  65
      BIGG=P02-APR*E0*RHO0-T1*C3                                             SU10  66
      CHECK=0                                                                SU10  67
      GO TO 725                                                              SU10  68
720   T1=APR+BPR/EE                                                          SU10  69
      T2=(BPR*E0)/(ESTAR*ETA**2*EE**2)                                       SU10  70
      BIGG=P02-T1*E0*RHO0-BIGAPR*MU-BIGBPR*MU**2                             SU10  71
      PGRHO=T1*E0+BIGAPR/RHOSTR+2.*BIGBPR*MU/RHOSTR+T2*2.*E0                 SU10  72
      PGRHO=-PGRHO                                                           SU10  73
      PGE=T1*RHO0-RHO0*T2                                                    SU10  74
      PGE=-PGE                                                               SU10  75
      CHECK=1                                                                SU10  76
725   CONTINUE                                                               SU10  77
870   CONTINUE                                                               SU10  78
      RETURN                                                                 SU10  79
      END                                                                    SU10  80
      SUBROUTINE SOUT2                                                       SU11   1
C                                                                            SU11   2
C         PRINTS 6 LINES OF DISCONTINUITY AT TO                              SU11   3
C                                                                            SU11   4
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPSSU11   5
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LENSU11   6
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS        SU11   7
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2SU11   8
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(SU11   9
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     115,4),RPART(15,2)                                                    SU11  10
C                                                                          SU11  11
C                                                                          SU11  12
      COMMON ZO,RO,PO,UO,VO,LO,MO,RHOO,EO,AO,UBARO,VBARO                   SU11  13
C                                                                          SU11  14
      COMMON NP,NT,NR,NI,NDEL,ISUB                                         SU11  15
C                                                                          SU11  16
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                      SU11  17
      COMMON DIRCOS                                                        SU11  18
      COMMON TIME                                                          SU11  19
      COMMON IRARF                                                         SU11  20
      COMMON KSTOP                                                         SU11  21
      COMMON TPSI                                                          SU11  22
      COMMON KKK                                                           SU11  23
      REAL LO,MO,LENGTH,MU,KO                                              SU11  24
C                                                                          SU11  25
4     FORMAT (///30H CURVES OF DISCONTINUITY AT TO//40X20H--PROJECTILE SSU11  26
     1HOCK--//)                                                           SU11  27
6     FORMAT (//35X29H--PROJECTILE PARTICLE CURVE--//)                     SU11  28
8     FORMAT (//42X16H--TARGET SHOCK--//)                                  SU11  29
10    FORMAT (//38X25H--TARGET PARTICLE CURVE--//)                         SU11  30
12    FORMAT (//42X15H--RAREFACTION--//)                                   SU11  31
14    FORMAT (//35X30H--RAREFACTION PARTICLE CURVE--//)                    SU11  32
16    FORMAT (7X1HZ19X1HR19X1HP19X1HU19X1HV/7X3HRHO17X1HE19X1HA19X1HL19XSU11  33
     11HM//)                                                              SU11  34
18    FORMAT (7X1HZ19X1HR//)                                               SU11  35
20    FORMAT (7X1HZ19X1HR19X1HL19X1HM//)                                   SU11  36
30    FORMAT (5E20.8/5E20.8//)                                             SU11  37
35    FORMAT (2E20.8)                                                      SU11  38
38    FORMAT (4E20.8)                                                      SU11  39
39    FORMAT (//35X16H--FREE SURFACE--//)                                  SU11  40
40    FORMAT (1H1)                                                         SU11  41
      WRITE (3,4)                                                          SU11  42
      WRITE (3,16)                                                         SU11  43
      WRITE (3,30) ((TAB2(I,J,1),J=1,10),I=1,NP)                           SU11  44
      WRITE (3,6)                                                          SU11  45
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      WRITE (3,18)                                                      SU11  46
      WRITE (3,35) ((SPART(I,J,1),J=1,2),I=1,NP)                        SU11  47
      WRITE (3,8)                                                       SU11  48
      WRITE (3,16)                                                      SU11  49
      WRITE (3,30) ((TAB2(I,J,2),J=1,10),I=1,NT)                        SU11  50
      WRITE (3,10)                                                      SU11  51
      WRITE (3,18)                                                      SU11  52
      WRITE (3,35) ((SPART(I,J,2),J=1,2),I=1,NT)                        SU11  53
      WRITE (3,12)                                                      SU11  54
      WRITE (3,20)                                                      SU11  55
      WRITE (3,38) ((RARF2(I,J),J=1,4),I=1,NR)                          SU11  56
      WRITE (3,14)                                                      SU11  57
      WRITE (3,39)                                                      SU11  58
      WRITE (3,35) ((SURF2(I,J),J=1,2),I=1,NP)                          SU11  59
      WRITE (3,40)                                                      SU11  60
      RETURN                                                            SU11  61
      END                                                               SU11  62
      SUBROUTINE SOUT                                                   SU12   1
C                                                                       SU12   2
C       PRINTS 4 LINES OF DISCONTINUITY AT TO-H                         SU12   3
C                                                                       SU12   4
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPSSU12   5
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LENSU12   6
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS       SU12   7
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2SU12   8
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(SU12   9
     115,4),RPART(15,2)                                                 SU12  10
C                                                                       SU12  11
C                                                                       SU12  12
      COMMON ZO,RO,PO,UO,VO,LO,MO,RHOO,EO,AO,UBARO,VBARO                SU12  13
C                                                                       SU12  14
      COMMON NP,NT,NR,NI,NDEL,ISUB                                      SU12  15
C                                                                       SU12  16
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                   SU12  17
      COMMON DIRCOS                                                     SU12  18
      COMMON TIME                                                       SU12  19
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      COMMON IRARF                                                      SU12  20
      COMMON KSTOP                                                      SU12  21
      COMMON TPSI                                                       SU12  22
      COMMON KKK                                                        SU12  23
      REAL L0,M0,LENGTH,MU,K0                                           SU12  24
C                                                                       SU12  25
4     FORMAT (///32H CURVES OF DISCONTINUITY AT TO-H//40X20H--PROJECTILESU12  26
     1 SHOCK--//)                                                       SU12  27
6     FORMAT (//42X16H--TARGET SHOCK--//)                               SU12  28
8     FORMAT (//42X15H--RAREFACTION--//)                                SU12  29
10    FORMAT (7X1HZ19X1HR19X1HP19X1HU19X1HV/7X3HRHO17X1HE19X1HA19X1HL19XSU12  30
     11HM//)                                                            SU12  31
15    FORMAT (5E20.8/5E20.8//)                                          SU12  32
18    FORMAT (1H1)                                                      SU12  33
21    FORMAT (//35X16H--FREE SURFACE--//)                               SU12  34
25    FORMAT (2E20.8)                                                   SU12  35
      WRITE (3,4)                                                       SU12  36
      WRITE (3,10)                                                      SU12  37
      WRITE (3,15) ((TAB(I,J,1),J=1,10),I=1,NP)                         SU12  38
      WRITE (3,6)                                                       SU12  39
      WRITE (3,10)                                                      SU12  40
      WRITE (3,15) ((TAB(I,J,2),J=1,10),I=1,NT)                         SU12  41
      WRITE (3,8)                                                       SU12  42
      WRITE (3,10)                                                      SU12  43
      WRITE (3,15) ((RARF(I,J),J=1,10),I=1,NR)                          SU12  44
      WRITE (3,21)                                                      SU12  45
      WRITE (3,25) ((SURF(I,J),J=1,2),I=1,NP)                           SU12  46
      WRITE (3,18)                                                      SU12  47
      RETURN                                                            SU12  48
      END                                                               SU12  49
      SUBROUTINE PRINT(BL,ZTAB,RTAB,KK)                                 SU13   1
C         PRINTS INTERIOR REGION                                        SU13   2
C                                                                       SU13   3
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPSSU13   4
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LENSU13   5
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS       SU13   6
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      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2SU13    7
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(SU13    8
     115,4),RPART(15,2)                                                SU13    9
C                                                                       SU13   10
C                                                                       SU13   11
      COMMON Z0,R0,P0,U0,V0,L0,M0,RHO0,E0,A0,UBAR0,VBAR0                SU13   12
C                                                                       SU13   13
      COMMON NP,NT,NR,NI,NDEL,ISUB                                      SU13   14
C                                                                       SU13   15
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                   SU13   16
      COMMON DIRCOS                                                     SU13   17
      COMMON TIME                                                       SU13   18
      COMMON IRARF                                                      SU13   19
      COMMON KSTOP                                                      SU13   20
      COMMON TPSI                                                       SU13   21
      COMMON KKK                                                        SU13   22
      REAL L0,M0,LENGTH,MU,K0                                           SU13   23
C                                                                       SU13   24
      DIMENSION BL(20,20,6),ZTAB(20),RTAB(20)                           SU13   25
C                                                                       SU13   26
      DO 15 I=1,20                                                      SU13   27
      DO 15 J=1,20                                                      SU13   28
      IF (ABS(BL(I,J,1))+ABS(BL(I,J,2))+ABS(BL(I,J,3)))  20,15,20       SU13   29
15    CONTINUE                                                          SU13   30
      WRITE (3,18)                                                      SU13   31
18    FORMAT (15H1TABLES ALL = 0/1H1)                                   SU13   32
      CALL EXIT                                                         SU13   33
20    I1=I                                                              SU13   34
      DO 30 I=I1,20                                                     SU13   35
      IF (ABS(BL(I,1,1))+ABS(BL(I,1,2))+ABS(BL(I,1,3)))  30,22,30       SU13   36
22    DO 25 J=1,20                                                      SU13   37
      IF (ABS(BL(I,J,1))+ABS(BL(I,J,2))+ABS(BL(I,J,3)))  30,25,30       SU13   38
25    CONTINUE                                                          SU13   39
      I2=I-1                                                            SU13   40
      GO TO 35                                                          SU13   41
30    CONTINUE                                                          SU13   42
```

```
      I2=20                                                              SU13   43
35    DO 45 J=1,20                                                       SU13   44
      IF (ABS(BL(I1,J,1))+ABS(BL(I1,J,2))+ABS(BL(I1,J,3)))  45,37,45     SU13   45
37    DO 40 I=I1,I2                                                      SU13   46
      IF (ABS(BL(I,J,1))+ABS(BL(I,J,2))+ABS(BL(I,J,3)))  45,40,45        SU13   47
40    CONTINUE                                                           SU13   48
      J2=J-1                                                             SU13   49
      GO TO 50                                                           SU13   50
45    CONTINUE                                                           SU13   51
      J2=20                                                              SU13   52
50    J1=1                                                               SU13   53
C                                                                        SU13   54
C         PRINT TABLE                                                    SU13   55
C                                                                        SU13   56
      GO TO (52,56),KK                                                   SU13   57
52    WRITE (3,53)                                                       SU13   58
53    FORMAT (//24HOINTERIOR REGION AT TO-H///)                          SU13   59
      GO TO 62                                                           SU13   60
56    WRITE (3,57)                                                       SU13   61
57    FORMAT (//22HOINTERIOR REGION AT TO///)                            SU13   62
62    DO 70 I=I1,I2                                                      SU13   63
      WRITE (3,64) ZTAB(I)                                               SU13   64
64    FORMAT (///7HOZTAB =,F10.4//7X1HR9X1HP17X1HU17X1HV17X3HRHO15X1HE17SU13   65
     1X1HA//)                                                            SU13   66
      DO 69 J=J1,J2                                                      SU13   67
      WRITE (3,68) RTAB(J),(BL(I,J,K),K=1,6)                             SU13   68
68    FORMAT (F12.4,6E18.8)                                              SU13   69
69    CONTINUE                                                           SU13   70
70    CONTINUE                                                           SU13   71
80    WRITE (3,82)                                                       SU13   72
82    FORMAT (1H1)                                                       SU13   73
      RETURN                                                             SU13   74
      END                                                                SU13   75
      SUBROUTINE NRIT2(X,Y,XO,DX,YO,DY,EX,EY,FGOF,IT,KODE)               SU14    1
C                                                                        SU14    2
C         NEWTON-RAPHSON METHOD FOR SOLUTION OF                          SU14    3
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C          TWO NON LINEAR EQUATIONS IN TWO UNKNOWNS                      SU14    4
C                                                                        SU14    5
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPSSU14    6
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LENSU14    7
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS       SU14    8
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2SU14    9
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(SU14   10
     115,4),RPART(15,2)                                                 SU14   11
C                                                                        SU14   12
C                                                                        SU14   13
      COMMON Z0,R0,P0,U0,V0,L0,M0,RHO0,E0,A0,UBAR0,VBAR0                 SU14   14
C                                                                        SU14   15
      COMMON NP,NT,NR,NI,NDEL,ISUB                                       SU14   16
C                                                                        SU14   17
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                    SU14   18
      COMMON DIRCOS                                                      SU14   19
      COMMON TIME                                                        SU14   20
      COMMON IRARF                                                       SU14   21
      COMMON KSTOP                                                       SU14   22
      COMMON TPSI                                                        SU14   23
      COMMON KKK                                                         SU14   24
      REAL L0,M0,LENGTH,MU,K0                                            SU14   25
C                                                                        SU14   26
      XB=X0                                                              SU14   27
      YB=Y0                                                              SU14   28
      DXX=DX                                                             SU14   29
      DYY=DY                                                             SU14   30
      DELX1=0                                                            SU14   31
      DELY1=0                                                            SU14   32
      KODE=0                                                             SU14   33
1     CONTINUE                                                           SU14   34
      DO 50 I=1,IT                                                       SU14   35
      KK=0                                                               SU14   36
      XX=XB+DXX                                                          SU14   37
      YY=YB+DYY                                                          SU14   38
C                                                                        SU14   39
```

```
C                                                                       SU14  40
      CALL FGOF(XB,YY,F2,G2)                                            SU14  41
      CALL FGOF(XX,YB,F1,G1)                                            SU14  42
      CALL FGOF(XB,YB,F0,G0)                                            SU14  43
      A1=(F1-F0)/DXX                                                    SU14  44
      B1=(F2-F0)/DYY                                                    SU14  45
      C1=-F0                                                            SU14  46
      A2=(G1-G0)/DXX                                                    SU14  47
      B2=(G2-G0)/DYY                                                    SU14  48
      C2=-G0                                                            SU14  49
      DET=A1*B2-A2*B1                                                   SU14  50
      IF (DET.EQ.0.) GO TO 920                                          SU14  51
      DELX=(B2*C1-B1*C2)/DET                                            SU14  52
      DELY=(A1*C2-A2*C1)/DET                                            SU14  53
      IF (ABS(DELX).GT..001) GO TO 8                                    SU14  54
      DELX=0.                                                           SU14  55
8     IF (ABS(DELY).GT..001) GO TO 9                                    SU14  56
      DELY=0.                                                           SU14  57
9     CONTINUE                                                          SU14  58
      SDEL=ABS(DELX+DELX1)+ABS(DELY+DELY1)                              SU14  59
      DELX1=DELX                                                        SU14  60
      DELY1=DELY                                                        SU14  61
C                                                                       SU14  62
C          TEST FOR SAME REGION                                         SU14  63
C                                                                       SU14  64
      DO 10 J=1,IT                                                      SU14  65
      XB1=XB+DELX                                                       SU14  66
      YB1=YB+DELY                                                       SU14  67
      IF (YB1.LE.0.)YB1=0.                                              SU14  68
      M=COMP(XB,YB,XB1,YB1)                                             SU14  69
      IF (M.EQ.1) GO TO 15                                              SU14  70
11    CONTINUE                                                          SU14  71
      KK=1                                                              SU14  72
      DELX=.5*DELX                                                      SU14  73
      DELY=.5*DELY                                                      SU14  74
10    CONTINUE                                                          SU14  75
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      GO TO 930                                                          SU14  76
15    IF (ABS(XB-XB1).GT.EX) GO TO 45                                    SU14  77
      IF (ABS(YB-YB1).GT.EY) GO TO 45                                    SU14  78
      IF (KK.NE.0) GO TO 45                                              SU14  79
      X=XB1                                                              SU14  80
      Y=YB1                                                              SU14  81
      KODE=0                                                             SU14  82
      RETURN                                                             SU14  83
45    CONTINUE                                                           SU14  84
      IF (KODE.NE.1) GO TO 46                                            SU14  85
      IF (SDEL.GT.EPI1) GO TO 46                                         SU14  86
      DELX=.5*DELX                                                       SU14  87
      DELY=.5*DELY                                                       SU14  88
      GO TO 9                                                            SU14  89
46    XB=XB1                                                             SU14  90
      YB=YB1                                                             SU14  91
      DEL=DELTA                                                          SU14  92
      DO 70 N=1,NDEL                                                     SU14  93
      XB2=XB+DEL                                                         SU14  94
      M=COMP(XB,YB,XB2,YB)                                               SU14  95
      IF (M.NE.1) GO TO 55                                               SU14  96
      DXX=DEL                                                            SU14  97
      GO TO 80                                                           SU14  98
55    XB2=XB-DEL                                                         SU14  99
      M=COMP(XB,YB,XB2,YB)                                               SU14 100
      IF (M.NE.1) GO TO 60                                               SU14 101
      DXX=-DEL                                                           SU14 102
      GO TO 80                                                           SU14 103
60    DEL=.5*DEL                                                         SU14 104
70    CONTINUE                                                           SU14 105
      GO TO 980                                                          SU14 106
80    DEL=DELTA                                                          SU14 107
      DO 100 N=1,NDEL                                                    SU14 108
      YB2=YB+DEL                                                         SU14 109
      M=COMP(XB,YB,XB,YB2)                                               SU14 110
      IF (M.NE.1) GO TO 85                                               SU14 111
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      DYY=DEL                                                                SU14  112
      GO TO 50                                                               SU14  113
85    YB2=YB-DEL                                                             SU14  114
      M=COMP(XB,YB,XB,YB2)                                                   SU14  115
      IF (M.NE.1) GO TO 90                                                   SU14  116
      DYY=-DEL                                                               SU14  117
      GO TO 50                                                               SU14  118
90    DEL=.5*DEL                                                             SU14  119
100   CONTINUE                                                               SU14  120
      GO TO 990                                                              SU14  121
50    CONTINUE                                                               SU14  122
      X=XB1                                                                  SU14  123
      Y=YB1                                                                  SU14  124
      IF (KODE.EQ.1) RETURN                                                  SU14  125
      KODE=1                                                                 SU14  126
      GO TO 1                                                                SU14  127
920   WRITE (3,922) I                                                        SU14  128
922   FORMAT (46HODETERMINANT IS 0 IN SUBR. NRIT2 FOR ITERATION,I4)          SU14  129
      GO TO 950                                                              SU14  130
930   KODE=2                                                                 SU14  131
      RETURN                                                                 SU14  132
950   WRITE (3,952) XO,YO,XB,YB,XB1,YB1,DELX,DELY                            SU14  133
952   FORMAT (1X5HZO  =,E15.8,4X5HRO  =,E15.8/1X5HZB  =,E15.8,4X5HRB  =,     SU14  134
     1E15.8/1X5HZB1 =,E15.8,4X5HRB1 =,E15.8/1X5HDELZ=,E15.8,4X5HDELR=,E1SU14  135
     15.8/1H1)                                                               SU14  136
      CALL EXIT                                                              SU14  137
      RETURN                                                                 SU14  138
980   KODE=3                                                                 SU14  139
      RETURN                                                                 SU14  140
990   KODE=4                                                                 SU14  141
      RETURN                                                                 SU14  142
      END                                                                    SU14  143
      FUNCTION COMP(ZP,RP,ZP1,RP1)                                           SU15    1
C                                                                            SU15    2
C         DETERMINES IF 2 POINTS ARE IN THE SAME REGION                      SU15    3
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPSSU15    4
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     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LENSU15    5
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS         SU15    6
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2SU15    7
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(SU15    8
     115,4),RPART(15,2)                                                  SU15    9
C                                                                        SU15   10
      COMMON Z0,R0,P0,U0,V0,L0,M0,RHO0,E0,A0,UBAR0,VBAR0                 SU15   11
C                                                                        SU15   12
      COMMON NP,NT,NR,NI,NDEL,ISUB                                       SU15   13
C                                                                        SU15   14
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                    SU15   15
      COMMON DIRCOS                                                      SU15   16
      COMMON TIME                                                        SU15   17
      COMMON IRARF                                                       SU15   18
      COMMON KSTOP                                                       SU15   19
      COMMON TPSI                                                        SU15   20
      COMMON KKK                                                         SU15   21
      REAL L0,M0,LENGTH,MU,K0                                            SU15   22
      EPS=.0000001                                                       SU15   23
      IF (RP1.LT.0.) GO TO 80                                            SU15   24
      IF (ZP1.GE.0.) GO TO 4                                             SU15   25
      IF (RP1.GT.RADIUS) GO TO 80                                        SU15   26
C                                                                        SU15   27
C         FIND CONTROL CONSTANTS FOR ZP,RP                               SU15   28
C                                                                        SU15   29
4     CONTINUE                                                           SU15   30
      IF (ITS3.EQ.1) GO TO 33                                            SU15   31
      IF (IRARF.EQ.1) GO TO 13                                           SU15   32
      M=PICK(ZP,RP,3)                                                    SU15   33
      IF (ZP.GT.RARF(M,1).AND.M.NE.1)M=M-1                               SU15   34
      FF=RP-RARF(M+1,2)-(RARF(M,2)-RARF(M+1,2))*(ZP-RARF(M+1,1))/(RARF(MSU15   35
     1,1)-RARF(M+1,1))                                                   SU15   36
      IF (FF.LT.0.) GO TO 11                                             SU15   37
10    NN=1                                                               SU15   38
      GO TO 13                                                           SU15   39
11    NN=0                                                               SU15   40
```

```
13    CONTINUE                                                              SU15  41
      DO 22 K=1,2                                                           SU15  42
      M=PICK(ZP1,RP1,K)                                                     SU15  43
      IF (RP1.LT.TAB(M,2,K).AND.M.NE.1)M=M-1                                SU15  44
5     CONTINUE                                                              SU15  45
      FF=ZP1-TAB(M+1,1,K)-(TAB(M,1,K)-TAB(M+1,1,K))*(RP1-TAB(M+1,2,K))/(SU15  46
     1TAB(M,2,K)-TAB(M+1,2,K))                                              SU15  47
      IF (K.EQ.2) GO TO 17                                                  SU15  48
      IF (RP1.GT.RADIUS) GO TO 21                                           SU15  49
      IF (FF.LT.-EPS) GO TO 90                                              SU15  50
      GO TO 21                                                              SU15  51
17    IF (FF.GT.EPS) GO TO 100                                              SU15  52
21    IF (RP.GT.RADIUS) GO TO 22                                            SU15  53
      M=PICK(ZP1,RP1,4)                                                     SU15  54
      FF=ZP1-SURF(M+1,1)-(SURF(M,1)-SURF(M+1,1))*(RP1-SURF(M+1,2))/(SURFSU15  55
     1(M,2)-SURF(M+1,2))                                                    SU15  56
      IF (FF)  80,22,22                                                     SU15  57
22    CONTINUE                                                              SU15  58
      IF (IRARF.EQ.1) GO TO 33                                              SU15  59
      M=PICK(ZP1,RP1,3)                                                     SU15  60
      IF (ZP1.GT.RARF(M,1).AND.M.NE.1)M=M-1                                 SU15  61
      FF=RP1-RARF(M+1,2)-(RARF(M,2)-RARF(M+1,2))*(ZP1-RARF(M+1,1))/(RARFSU15  62
     1(M,1)-RARF(M+1,1))                                                    SU15  63
      IF (FF.LT.0.) GO TO 31                                                SU15  64
30    NN1=1                                                                 SU15  65
      GO TO 32                                                              SU15  66
31    NN1=0                                                                 SU15  67
32    IF (NN1.NE.NN) GO TO 110                                              SU15  68
33    CONTINUE                                                              SU15  69
      COMP=1                                                                SU15  70
      COMP=COMP+.2                                                          SU15  71
      RETURN                                                                SU15  72
80    COMP=2                                                                SU15  73
      COMP=COMP+.2                                                          SU15  74
      RETURN                                                                SU15  75
90    COMP=3.                                                               SU15  76
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      COMP=COMP+.2                                                      SU15  77
      RETURN                                                            SU15  78
100   COMP=4.                                                           SU15  79
      COMP=COMP+.2                                                      SU15  80
      RETURN                                                            SU15  81
110   COMP=5.                                                           SU15  82
      COMP=COMP+.2                                                      SU15  83
      RETURN                                                            SU15  84
      END                                                               SU15  85
      FUNCTION PICK(ZP,RP,KODE)                                         SU16   1
C                                                                       SU16   2
C         DETERMINES 2 CLOSEST CONSECUTIVE POINTS ON SPECIFIED          SU16   3
C         LINE OF DISCONTINUITY TO A GIVEN POINT                        SU16   4
C                                                                       SU16   5
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPSSU16   6
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LENSU16   7
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS       SU16   8
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2SU16   9
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(SU16  10
     115,4),RPART(15,2)                                                 SU16  11
C                                                                       SU16  12
C                                                                       SU16  13
      COMMON ZO,RO,PO,UO,VO,LO,MO,RHOO,EO,AO,UBARO,VBARO                SU16  14
C                                                                       SU16  15
      COMMON NP,NT,NR,NI,NDEL,ISUB                                      SU16  16
C                                                                       SU16  17
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                   SU16  18
      COMMON DIRCOS                                                     SU16  19
      COMMON TIME                                                       SU16  20
      COMMON IRARF                                                      SU16  21
      COMMON KSTOP                                                      SU16  22
      COMMON TPSI                                                       SU16  23
      COMMON KKK                                                        SU16  24
      REAL LO,MO,LENGTH,MU,KO                                           SU16  25
C                                                                       SU16  26
      GO TO (5,10,100,300),KODE                                         SU16  27
```

```
5       NN=NP                                                          SU16  28
        K=1                                                            SU16  29
        GO TO 15                                                       SU16  30
10      NN=NT                                                          SU16  31
        K=2                                                            SU16  32
15      AA=(TAB(1,1,K)-ZP)**2+(TAB(1,2,K)-RP)**2                       SU16  33
C                                                                      SU16  34
C           SEARCH SHOCK TABLES                                        SU16  35
C                                                                      SU16  36
        DO 60 N=2,NN                                                   SU16  37
        A=(TAB(N,1,K)-ZP)**2+(TAB(N,2,K)-RP)**2                        SU16  38
        IF (A.GE.AA) GO TO 23                                          SU16  39
        AA=A                                                           SU16  40
60      CONTINUE                                                       SU16  41
        PICK=NN-1                                                      SU16  42
        PICK=PICK+.2                                                   SU16  43
        RETURN                                                         SU16  44
23      PICK=N-1                                                       SU16  45
        PICK=PICK+.2                                                   SU16  46
        RETURN                                                         SU16  47
100     AA=(RARF(1,1)-ZP)**2+(RARF(1,2)-RP)**2                         SU16  48
C                                                                      SU16  49
C           SEARCH RAREFACTION TABLE                                   SU16  50
C                                                                      SU16  51
        DO 200 N=2,NR                                                  SU16  52
        A=(RARF(N,1)-ZP)**2+(RARF(N,2)-RP)**2                          SU16  53
        IF (A.GE.AA) GO TO 203                                         SU16  54
        AA=A                                                           SU16  55
200     CONTINUE                                                       SU16  56
        PICK=NR-1                                                      SU16  57
        PICK=PICK+.2                                                   SU16  58
        RETURN                                                         SU16  59
203     PICK=N-1                                                       SU16  60
        PICK=PICK+.2                                                   SU16  61
        RETURN                                                         SU16  62
300     AA=(SURF(1,1)-ZP)**2+(SURF(1,2)-RP)**2                         SU16  63
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C                                                                       SU16  64
C     SEARCH FREE SURFACE TABLE                                         SU16  65
C                                                                       SU16  66
      DO 400 N=2,NP                                                     SU16  67
      A=(SURF(N,1)-ZP)**2+(SURF(N,2)-RP)**2                             SU16  68
      IF (A.GE.AA) GO TO 303                                            SU16  69
      AA=A                                                              SU16  70
400   CONTINUE                                                          SU16  71
      PICK=NP-1                                                         SU16  72
      PICK=PICK+.2                                                      SU16  73
      RETURN                                                            SU16  74
303   PICK=N-1                                                          SU16  75
      PICK=PICK+.2                                                      SU16  76
304   CONTINUE                                                          SU16  77
      RETURN                                                            SU16  78
      END                                                               SU16  79
      SUBROUTINE GUESS(KOD1,KOD2,ZP,RP,I2,K2,ZG,RG,DZ,DR)               SU17   1
C                                                                       SU17   2
C         DETERMINES STARTING POINT AND DELTAS                          SU17   3
C         FOR NEWTON-RAPHSON ITERATION                                  SU17   4
C                                                                       SU17   5
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPSSU17   6
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LENSU17   7
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS       SU17   8
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2SU17   9
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(SU17  10
     115,4),RPART(15,2)                                                 SU17  11
C                                                                       SU17  12
C                                                                       SU17  13
      COMMON ZO,RO,PO,UO,VO,LO,MO,RHOO,EO,AO,UBARO,VBARO                SU17  14
C                                                                       SU17  15
      COMMON NP,NT,NR,NI,NDEL,ISUB                                      SU17  16
C                                                                       SU17  17
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                   SU17  18
      COMMON DIRCOS                                                     SU17  19
      COMMON TIME                                                       SU17  20
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      COMMON IRARF                                                      SU17  21
      COMMON KSTOP                                                      SU17  22
      COMMON TPSI                                                       SU17  23
      COMMON KKK                                                        SU17  24
      REAL LO,MO,LENGTH,MU,KO                                           SU17  25
C                                                                       SU17  26
      KS=0                                                              SU17  27
      IF (KOD1.EQ.2) GO TO 10                                           SU17  28
      ZG=TAB(I2,1,K2)                                                   SU17  29
      RG=TAB(I2,2,K2)                                                   SU17  30
      IF (IRARF.EQ.1) GO TO 9                                           SU17  31
2     M=1-(NR-2)*(K2-2)                                                 SU17  32
      FF=RG-RARF(M+1,2)-(RARF(M,2)-RARF(M+1,2))*(ZG-RARF(M+1,1))/(RARF(MSU17  33
     1,1)-RARF(M+1,1))                                                  SU17  34
      IF (FF.GT.0.) GO TO 9                                             SU17  35
      ZG=(1.-.02/(RADIUS-RG))*ZG                                        SU17  36
      RG=RG+.02                                                         SU17  37
      IF (RG.GT.(RADIUS-.01)) GO TO 110                                 SU17  38
      GO TO 2                                                           SU17  39
9     CONTINUE                                                          SU17  40
      GO TO 50                                                          SU17  41
10    JJJ=NT-1                                                          SU17  42
      IF (ITS3.EQ.1) GO TO 26                                           SU17  43
      DO 24 M=1,JJJ                                                     SU17  44
      IF (RP.GT.TAB(M+1,2,2).OR.RP.LT.TAB(M,2,2)) GO TO 24              SU17  45
      GO TO 25                                                          SU17  46
24    CONTINUE                                                          SU17  47
25    CONTINUE                                                          SU17  48
      FF=ZP-TAB(M+1,1,2)-(TAB(M,1,2)-TAB(M+1,1,2))*(RP-TAB(M+1,2,2))/(TASU17  49
     1B(M,2,2)-TAB(M+1,2,2))                                            SU17  50
      IF (FF.GT.0.) GO TO 20                                            SU17  51
26    CONTINUE                                                          SU17  52
      ZG=ZP                                                             SU17  53
      RG=RP                                                             SU17  54
      GO TO 50                                                          SU17  55
20    IK=M                                                              SU17  56
```

```
      M=CROSS(TAB(IK,1,2),TAB(IK,2,2),TAB(IK+1,1,2),TAB(IK+1,2,2),0.,0.,SU17  57
     1ZP,RP,ZG,RG)                                                    SU17  58
      GO TO (50,920,930),M                                            SU17  59
C                                                                     SU17  60
C         COMPUTE DELTAS                                              SU17  61
C                                                                     SU17  62
50    DEL=DELTA                                                       SU17  63
C                                                                     SU17  64
C         Z DELTA                                                     SU17  65
C                                                                     SU17  66
      LL=0                                                            SU17  67
52    DO 70 N=1,NDEL                                                  SU17  68
      ZZ=ZG+DEL                                                       SU17  69
      M=COMP(ZG,RG,ZZ,RG)                                             SU17  70
      IF (M.NE.1) GO TO 55                                            SU17  71
      DZ=DEL                                                          SU17  72
      GO TO 80                                                        SU17  73
55    ZZ=ZG-DEL                                                       SU17  74
      M=COMP(ZG,RG,ZZ,RG)                                             SU17  75
      IF (M.NE.1) GO TO 60                                            SU17  76
      DZ=-DEL                                                         SU17  77
      GO TO 80                                                        SU17  78
60    DEL=.5*DEL                                                      SU17  79
70    CONTINUE                                                        SU17  80
      LL=LL+1                                                         SU17  81
      IF (LL.EQ.3) GO TO 75                                           SU17  82
      RG=RG-DELTA/5.                                                  SU17  83
      DEL=DELTA                                                       SU17  84
      GO TO 52                                                        SU17  85
75    KOD2=2                                                          SU17  86
      RETURN                                                          SU17  87
80    DEL=DELTA                                                       SU17  88
      IF (KS.EQ.1) GO TO 120                                          SU17  89
C                                                                     SU17  90
C         R DELTA                                                     SU17  91
C                                                                     SU17  92
```

```
      LL=0                                                              SU17  93
82    DO 100 N=1,NDEL                                                   SU17  94
      RR=RG+DEL                                                         SU17  95
      M=COMP(ZG,RG,ZG,RR)                                               SU17  96
      IF (M.NE.1) GO TO 85                                              SU17  97
      DR=DEL                                                            SU17  98
      IF (KS.EQ.1) GO TO 50                                             SU17  99
      GO TO 120                                                         SU17 100
85    RR=RG-DEL                                                         SU17 101
      M=COMP(ZG,RG,ZG,RR)                                               SU17 102
      IF (M.NE.1) GO TO 90                                              SU17 103
      DR=-DEL                                                           SU17 104
      IF (KS.EQ.1) GO TO 50                                             SU17 105
      GO TO 120                                                         SU17 106
90    DEL=.5*DEL                                                        SU17 107
100   CONTINUE                                                          SU17 108
108   CONTINUE                                                          SU17 109
      LL=LL+1                                                           SU17 110
      FLL=LL                                                            SU17 111
      IF (LL.EQ.5) GO TO 110                                            SU17 112
C                                                                       SU17 113
104   ZG1=ZG+DELTA/5.*FLL*(-1.)**LL                                     SU17 114
      M=COMP(ZG,RG,ZG1,RG)                                              SU17 115
      ZG=ZG1                                                            SU17 116
      IF (M.NE.1) GO TO 108                                             SU17 117
      KS=1                                                              SU17 118
      GO TO 82                                                          SU17 119
110   KOD2=2                                                            SU17 120
      RETURN                                                            SU17 121
120   KOD2=1                                                            SU17 122
      RETURN                                                            SU17 123
920   WRITE (3,922)                                                     SU17 124
922   FORMAT (42HOERROR FOR COINCIDENT LINES IN SUBR. GUESS)            SU17 125
      GO TO 950                                                         SU17 126
C                                                                       SU17 127
930   WRITE (3,932)                                                     SU17 128
```

```
932   FORMAT (40HOERROR FOR PARALLEL LINES IN SUBR. GUESS)                SU17 129
C                                                                         SU17 130
950   WRITE (3,952) KOD1,I2,K2,ZP,RP                                      SU17 131
952   FORMAT (1X5HKOD1=,I4,4X3HI2=,I4,4X3HK2=,I4/1X3HZP=,E15.8,4X3HRP=,ESU17 132
     115.8/1H1)                                                           SU17 133
      XYZ=-2.                                                             SU17 134
      ZYX=SQRT(XYZ)                                                       SU17 135
      CALL EXIT                                                           SU17 136
      RETURN                                                              SU17 137
      END                                                                 SU17 138
      FUNCTION CROSS(X1,Y1,X2,Y2,X3,Y3,X4,Y4,X,Y)                         SU18   1
C                                                                         SU18   2
C         FINDS INTERSECTION OF TWO STRAIGHT LINES                        SU18   3
C                                                                         SU18   4
      EPS=.0000001                                                        SU18   5
      A1=Y2-Y1                                                            SU18   6
      B1=X1-X2                                                            SU18   7
      C1=X1*A1+Y1*B1                                                      SU18   8
      A2=Y4-Y3                                                            SU18   9
      B2=X3-X4                                                            SU18  10
      C2=X3*A2+Y3*B2                                                      SU18  11
      DET=A1*B2-A2*B1                                                     SU18  12
      D1=C1*B2-C2*B1                                                      SU18  13
      D2=A1*C2-A2*C1                                                      SU18  14
      IF (ABS(DET).LE.EPS) GO TO 10                                       SU18  15
      X=D1/DET                                                            SU18  16
      Y=D2/DET                                                            SU18  17
      CROSS=1                                                             SU18  18
      CROSS=CROSS+.2                                                      SU18  19
      RETURN                                                              SU18  20
10    IF (ABS(D1).GT.EPS) GO TO 20                                        SU18  21
      IF (ABS(D2).LE.EPS) GO TO 30                                        SU18  22
20    CROSS=3                                                             SU18  23
      CROSS=CROSS+.2                                                      SU18  24
      RETURN                                                              SU18  25
30    X=X1                                                                SU18  26
```

```
      Y=Y1                                                              SU18  27
      CROSS=2                                                           SU18  28
      CROSS=CROSS+.2                                                    SU18  29
316   CONTINUE                                                          SU18  30
      RETURN                                                            SU18  31
      END                                                               SU18  32
      FUNCTION PART(MODE,ZP,RP,ZX,RX,DELTA,NDEL)                        SU19   1
C                                                                       SU19   2
C         LOCATES A POINT IN THE SAME REGION AS A GIVEN POINT           SU19   3
C         TO BE USED IN COMPUTING A PARTIAL                             SU19   4
C     MODE=1,WITH RESPECT TO R                                          SU19   5
C     MODE=2,WITH RESPECT TO Z                                          SU19   6
C                                                                       SU19   7
      GO TO (2,4),MODE                                                  SU19   8
2     DR=DELTA                                                          SU19   9
      DZ=0.                                                             SU19  10
      GO TO 8                                                           SU19  11
4     DR=0.                                                             SU19  12
      DZ=DELTA                                                          SU19  13
8     DO 50 NN=1,NDEL                                                   SU19  14
      RR=RP+DR                                                          SU19  15
      ZZ=ZP+DZ                                                          SU19  16
      M=COMP(ZP,RP,ZZ,RR)                                               SU19  17
      IF (M.EQ.1) GO TO 60                                              SU19  18
      RR=RP-DR                                                          SU19  19
      ZZ=ZP-DZ                                                          SU19  20
      M=COMP(ZP,RP,ZZ,RR)                                               SU19  21
      IF (M.EQ.1) GO TO 60                                              SU19  22
      DZ=DZ*.5                                                          SU19  23
      DR=DR*.5                                                          SU19  24
50    CONTINUE                                                          SU19  25
      PART=2                                                            SU19  26
      PART=PART+.2                                                      SU19  27
      RETURN                                                            SU19  28
60    ZX=ZZ                                                             SU19  29
      RX=RR                                                             SU19  30
```

```
      PART=1                                                              SU19  31
      PART=PART+.2                                                        SU19  32
      RETURN                                                              SU19  33
      END                                                                 SU19  34
      FUNCTION PICK2(ZP,RP,KODE)                                          SU20   1
C                                                                         SU20   2
C         DETERMINES 2 CLOSEST CONSECUTIVE POINTS ON SPECIFIED            SU20   3
C         LINE OF DISCONTINUITY TO A GIVEN POINT                          SU20   4
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPSSU20   5
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LENSU20   6
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS         SU20   7
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2SU20   8
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(SU20   9
     115,4),RPART(15,2)                                                   SU20  10
C                                                                         SU20  11
      COMMON Z0,R0,P0,U0,V0,L0,M0,RHO0,E0,A0,UBAR0,VBAR0                  SU20  12
      COMMON NP,NT,NR,NI,NDEL,ISUB                                        SU20  13
C                                                                         SU20  14
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                     SU20  15
      COMMON DIRCOS                                                       SU20  16
      COMMON TIME                                                         SU20  17
      COMMON IRARF                                                        SU20  18
      COMMON KSTOP                                                        SU20  19
      COMMON TPSI                                                         SU20  20
      COMMON KKK                                                          SU20  21
      REAL L0,M0,LENGTH,MU,K0                                             SU20  22
C                                                                         SU20  23
      GO TO (5,10,100,205,210,300,500),KODE                               SU20  24
5     NN=NP                                                               SU20  25
      K=1                                                                 SU20  26
      GO TO 15                                                            SU20  27
10    NN=NT                                                               SU20  28
      K=2                                                                 SU20  29
15    AA=(TAB2(1,1,K)-ZP)**2+(TAB2(1,2,K)-RP)**2                          SU20  30
C                                                                         SU20  31
C               SEARCH SHOCK TABLES                                       SU20  32
```

```
C                                                                       SU20  33
      DO 60 N=2,NN                                                      SU20  34
      A=(TAB2(N,1,K)-ZP)**2+(TAB2(N,2,K)-RP)**2                         SU20  35
      IF (A.GE.AA) GO TO 23                                             SU20  36
      AA=A                                                              SU20  37
60    CONTINUE                                                          SU20  38
      PICK2=NN-1                                                        SU20  39
      PICK2=PICK2+.2                                                    SU20  40
      RETURN                                                            SU20  41
23    PICK2=N-1                                                         SU20  42
      PICK2=PICK2+.2                                                    SU20  43
      RETURN                                                            SU20  44
100   AA=(RARF2(1,1)-ZP)**2+(RARF2(1,2)-RP)**2                          SU20  45
C                                                                       SU20  46
C               SEARCH RAREFACTION TABLE                                SU20  47
C                                                                       SU20  48
      DO 200 N=2,NR                                                     SU20  49
      A=(RARF2(N,1)-ZP)**2+(RARF2(N,2)-RP)**2                           SU20  50
      IF (A.GE.AA) GO TO 203                                            SU20  51
      AA=A                                                              SU20  52
200   CONTINUE                                                          SU20  53
      PICK2=NR-1                                                        SU20  54
      PICK2=PICK2+.2                                                    SU20  55
      RETURN                                                            SU20  56
203   PICK2=N-1                                                         SU20  57
      PICK2=PICK2+.2                                                    SU20  58
      RETURN                                                            SU20  59
205   NN=NP                                                             SU20  60
      K=1                                                               SU20  61
      GO TO 215                                                         SU20  62
210   NN=NT                                                             SU20  63
      K=2                                                               SU20  64
215   AA=(SPART(1,1,K)-ZP)**2+(SPART(1,2,K)-RP)**2                      SU20  65
C                                                                       SU20  66
C               SEARCH SHOCK PARTICLE TABLES                            SU20  67
C                                                                       SU20  68
```

```
      DO 260 N=2,NN                                                    SU20  69
      A=(SPART(N,1,K)-ZP)**2+(SPART(N,2,K)-RP)**2                      SU20  70
      IF (A.GE.AA) GO TO 223                                           SU20  71
      AA=A                                                             SU20  72
260   CONTINUE                                                         SU20  73
      PICK2=NN-1                                                       SU20  74
      PICK2=PICK2+.2                                                   SU20  75
      RETURN                                                           SU20  76
223   PICK2=N-1                                                        SU20  77
      PICK2=PICK2+.2                                                   SU20  78
      RETURN                                                           SU20  79
300   AA=(RPART(1,1)-ZP)**2+(RPART(1,2)-RP)**2                         SU20  80
C                                                                      SU20  81
C              SEARCH RAREFACTION PARTICLE TABLE                       SU20  82
C                                                                      SU20  83
      DO 400 N=2,NR                                                    SU20  84
      A=(RPART(N,1)-ZP)**2+(RPART(N,2)-RP)**2                          SU20  85
      IF (A.GE.AA) GO TO 403                                           SU20  86
      AA=A                                                             SU20  87
400   CONTINUE                                                         SU20  88
      PICK2=NR-1                                                       SU20  89
      PICK2=PICK2+.2                                                   SU20  90
      RETURN                                                           SU20  91
403   PICK2=N-1                                                        SU20  92
      PICK2=PICK2+.2                                                   SU20  93
      RETURN                                                           SU20  94
500   AA=(SURF2(1,1)-ZP)**2+(SURF2(1,2)-RP)**2                         SU20  95
C                                                                      SU20  96
C     SEARCH FREE SURFACE TABLE                                        SU20  97
C                                                                      SU20  98
      DO 520 N=2,NP                                                    SU20  99
      A=(SURF2(N,1)-ZP)**2+(SURF2(N,2)-RP)**2                          SU20 100
      IF (A.GE.AA) GO TO 523                                           SU20 101
      AA=A                                                             SU20 102
520   CONTINUE                                                         SU20 103
      PICK2=NP-1                                                       SU20 104
```

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      PICK2=PICK2+.2                                                    SU20 105
      RETURN                                                            SU20 106
523   PICK2=N-1                                                         SU20 107
      PICK2=PICK2+.2                                                    SU20 108
      RETURN                                                            SU20 109
      END                                                               SU20 110
      FUNCTION TEST(ZP,RP)                                              SU21   1
C                                                                       SU21   2
C                                                                       SU21   3
C         DETERMINES IF A GIVEN INTERIOR POINT IS IN                    SU21   4
C         THE REGION TO BE CONSIDERED                                   SU21   5
C                                                                       SU21   6
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPSSU21   7
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LENSU21   8
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS       SU21   9
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2SU21  10
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(SU21  11
     115,4),RPART(15,2)                                                 SU21  12
C                                                                       SU21  13
C                                                                       SU21  14
      COMMON ZO,RO,PO,UO,VO,LO,MO,RHOO,EO,AO,UBARO,VBARO                SU21  15
C                                                                       SU21  16
      COMMON NP,NT,NR,NI,NDEL,ISUB                                      SU21  17
C                                                                       SU21  18
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                   SU21  19
      COMMON DIRCOS                                                     SU21  20
      COMMON TIME                                                       SU21  21
      COMMON IRARF                                                      SU21  22
      COMMON KSTOP                                                      SU21  23
      COMMON TPSI                                                       SU21  24
      COMMON KKK                                                        SU21  25
      REAL LO,MO,LENGTH,MU,KO                                           SU21  26
C                                                                       SU21  27
      EPS=.0001                                                         SU21  28
      IF (ITS3.EQ.1) GO TO 5                                            SU21  29
      IF (RP.GT.(RADIUS+EPS)) GO TO 5                                   SU21  30
```

```
      M=PICK2(ZP,RP,7)                                                  SU21   31
      FF=ZP-SURF2(M+1,1)-(SURF2(M,1)-SURF2(M+1,1))*(RP-SURF2(M,2))/(SURFSU21   32
     12(M,2)-SURF2(M+1,2))                                              SU21   33
      KICK=5                                                            SU21   34
      CALL DVCHK(KQ)                                                    SU21   35
      IF (KQ.EQ.1) GO TO 9980                                           SU21   36
      IF (FF)  200,5,5                                                  SU21   37
5     CONTINUE                                                          SU21   38
      IF (ZP.GT.-EPS) GO TO 1                                           SU21   39
      IF (RP.GT.(RADIUS+EPS)) GO TO 200                                 SU21   40
1     DO 10 K=1,2                                                       SU21   41
      IF (ITS3.EQ.1) GO TO 100                                          SU21   42
      M=PICK2(ZP,RP,K)                                                  SU21   43
      IF (TAB2(M,2,K).GT.RP.AND.M.NE.1)M=M-1                            SU21   44
      FF=ZP-TAB2(M+1,1,K)-(TAB2(M,1,K)-TAB2(M+1,1,K))*(RP-TAB2(M+1,2,K))SU21   45
     1/(TAB2(M,2,K)-TAB2(M+1,2,K))                                      SU21   46
      KICK=50                                                           SU21   47
      CALL DVCHK(KQ)                                                    SU21   48
      IF (KQ.EQ.1) GO TO 9980                                           SU21   49
      IF (K.EQ.2) GO TO 50                                              SU21   50
      IF (RP.GT.RADIUS) GO TO 10                                        SU21   51
      IF (FF)  200,10,10                                                SU21   52
50    IF (FF)  10,10,200                                                SU21   53
10    CONTINUE                                                          SU21   54
      DO 20 K=1,2                                                       SU21   55
      J=K+3                                                             SU21   56
      M=PICK2(ZP,RP,J)                                                  SU21   57
      IF (SPART(M,2,K).GT.RP.AND.M.NE.1)M=M-1                           SU21   58
      FF=ZP-SPART(M+1,1,K)-(SPART(M,1,K)-SPART(M+1,1,K))*(RP-SPART(M+1,2SU21   59
     1,K))/(SPART(M,2,K)-SPART(M+1,2,K))                                SU21   60
      KICK=15                                                           SU21   61
      CALL DVCHK(KQ)                                                    SU21   62
      IF (KQ.EQ.1) GO TO 9980                                           SU21   63
      IF (K.EQ.2) GO TO 15                                              SU21   64
      IF (RP.GT.RADIUS) GO TO 20                                        SU21   65
      IF (FF.LT..001) GO TO 300                                         SU21   66
```

```
      GO TO 20                                                                  SU21  67
15    IF (FF.GT.-.001) GO TO 400                                                SU21  68
20    CONTINUE                                                                  SU21  69
      IF (IRARF.EQ.1) GO TO 100                                                 SU21  70
      M=PICK2(ZP,RP,3)                                                          SU21  71
      FF=RP-RARF2(M+1,2)-(RARF2(M,2)-RARF2(M+1,2))*(ZP-RARF2(M+1,1))/(RASU21  72
     1RF2(M,1)-RARF2(M+1,1))                                                    SU21  73
      KICK=20                                                                   SU21  74
      CALL DVCHK(KQ)                                                            SU21  75
      IF (KQ.EQ.1) GO TO 9980                                                   SU21  76
      IF (FF.LT.0.) GO TO 100                                                   SU21  77
      M=PICK2(ZP,RP,6)                                                          SU21  78
      FF=RP-RPART(M+1,2)-(RPART(M,2)-RPART(M+1,2))*(ZP-RPART(M+1,1))/(RPSU21  79
     1ART(M,1)-RPART(M+1,1))                                                    SU21  80
      KICK=100                                                                  SU21  81
      CALL DVCHK(KQ)                                                            SU21  82
      IF (KQ.EQ.1) GO TO 9980                                                   SU21  83
      IF (FF.LT.0.) GO TO 500                                                   SU21  84
100   TEST=1                                                                    SU21  85
      TEST=TEST+.2                                                              SU21  86
      RETURN                                                                    SU21  87
200   TEST=2                                                                    SU21  88
      TEST=TEST+.2                                                              SU21  89
      RETURN                                                                    SU21  90
300   TEST=3                                                                    SU21  91
      TEST=TEST+.2                                                              SU21  92
      RETURN                                                                    SU21  93
400   TEST=4                                                                    SU21  94
      TEST=TEST+.2                                                              SU21  95
      RETURN                                                                    SU21  96
500   TEST=5                                                                    SU21  97
      TEST=TEST+.2                                                              SU21  98
      RETURN                                                                    SU21  99
9980  WRITE (3,9985) KICK                                                       SU21 100
9985  FORMAT (32HODIVIDE CHECK NEAR STATEMENT NO.,I5,14H IN SUBR. TEST/1SU21 101
     1H1)                                                                       SU21 102
```

```
      RETURN                                                            SU21  103
      END                                                               SU21  104
      SUBROUTINE FGOF5(Z5,R5,SS,QQ)                                     SU22    1
C                                                                       SU22    2
C         COMPUTES S5,Q5 FOR INTERIOR REGION                            SU22    3
C         ITERATION FOR Z5,R5                                           SU22    4
C                                                                       SU22    5
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPSSU22    6
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LENSU22    7
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS       SU22    8
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2SU22    9
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(SU22   10
     115,4),RPART(15,2)                                                 SU22   11
C                                                                       SU22   12
C                                                                       SU22   13
      COMMON Z0,R0,P0,U0,V0,L0,M0,RHO0,E0,A0,UBAR0,VBAR0                SU22   14
C                                                                       SU22   15
      COMMON NP,NT,NR,NI,NDEL,ISUB                                      SU22   16
C                                                                       SU22   17
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                   SU22   18
      COMMON DIRCOS                                                     SU22   19
      COMMON TIME                                                       SU22   20
      COMMON IRARF                                                      SU22   21
      COMMON KSTOP                                                      SU22   22
      COMMON TPSI                                                       SU22   23
      COMMON KKK                                                        SU22   24
      REAL L0,M0,LENGTH,MU,K0                                           SU22   25
C                                                                       SU22   26
      DIMENSION ANS(6)                                                  SU22   27
C                                                                       SU22   28
      CALL DBLTRP(Z5,R5,ANS)                                            SU22   29
      U5=ANS(2)                                                         SU22   30
      V5=ANS(3)                                                         SU22   31
      SS=Z5-Z0+H*V5                                                     SU22   32
      QQ=R5-R0+H*U5                                                     SU22   33
      RETURN                                                            SU22   34
```

```
      END                                                                SU22   35
      SUBROUTINE ITRP                                                    SU23    1
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPSSU23    2
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LENSU23    3
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS       SU23    4
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2SU23    5
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(SU23    6
     115,4),RPART(15,2)                                                 SU23    7
C                                                                       SU23    8
C                                                                       SU23    9
      COMMON Z0,R0,P0,U0,V0,L0,M0,RHO0,E0,A0,UBAR0,VBAR0                 SU23   10
C                                                                       SU23   11
      COMMON NP,NT,NR,NI,NDEL,ISUB                                      SU23   12
C                                                                       SU23   13
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                   SU23   14
      COMMON DIRCOS                                                     SU23   15
      COMMON TIME                                                       SU23   16
      COMMON IRARF                                                      SU23   17
      COMMON KSTOP                                                      SU23   18
      COMMON TPSI                                                       SU23   19
      COMMON KKK                                                        SU23   20
C                                                                       SU23   21
C                                                                       SU23   22
      REAL L0,M0,LENGTH,MU,K0                                           SU23   23
C                                                                       SU23   24
C  INTERPOLATION SCHEME FOR POINTS BETWEEN PARTICLE CURVES              SU23   25
C  AND DISCONTINUITIES                                                  SU23   26
C                                                                       SU23   27
      EPS=.0000001                                                      SU23   28
      KOD1=0                                                            SU23   29
      KOD2=0                                                            SU23   30
      DO 1000 J=1,20                                                    SU23   31
      DO 1000 I=1,20                                                    SU23   32
      M=TEST(Z(I),R(J))                                                 SU23   33
      IF (M.NE.5) GO TO 906                                             SU23   34
      IF (IRARF.EQ.1) GO TO 906                                         SU23   35
```

```
      NR1=NR-1                                                          SU23   36
      DO 907 JJ=1,NR1                                                   SU23   37
      IF (RARF2(JJ,1).LT.Z(I).AND.RARF2(JJ+1,1).GT.Z(I)) GO TO 908      SU23   38
907   CONTINUE                                                          SU23   39
908   CONTINUE                                                          SU23   40
      FF=R(J)-RARF2(JJ+1,2)+(RARF2(JJ,2)-RARF2(JJ+1,2))*(Z(I)-RARF2(JJ+1SU23   41
     1,1))/(RARF2(JJ,1)-RARF2(JJ+1,1))                                  SU23   42
      IF (FF.LT.0.) GO TO 1000                                          SU23   43
906   CONTINUE                                                          SU23   44
      IF (M.EQ.3.AND.Z(I).LT.EPS.AND.ABS(R(J)-RADIUS).LT.EPS) GO TO 1000SU23   45
      IF (M.LT.3) GO TO 8051                                            SU23   46
      WRITE (3,8050)                                                    SU23   47
8050  FORMAT (30HOINTERPOLATION SCHEME EMPLOYED)                        SU23   48
614   FORMAT (1X5HZ0  =,E15.8,4X5HR0  =,E15.8)                          SU23   49
      Z0=Z(I)                                                           SU23   50
      R0=R(J)                                                           SU23   51
      WRITE (3,614) Z0,R0                                               SU23   52
      WRITE (3,8888) M                                                  SU23   53
8888  FORMAT (1X4HM  =,I4)                                              SU23   54
8051  CONTINUE                                                          SU23   55
      GO TO (1000,1000,1001,1010,1100),M                                SU23   56
1001  IF (IRARF.EQ.1) GO TO 1003                                        SU23   57
      M=PICK2(Z(I),R(J),3)                                              SU23   58
      FF=R(J)-RARF2(M+1,2)+(RARF2(M,2)-RARF2(M+1,2))*(Z(I)-RARF2(M+1,1))SU23   59
     1/(RARF2(M,1)-RARF2(M+1,1))                                        SU23   60
      IF (FF.GT.0.) GO TO 1003                                          SU23   61
      DO 8002 K=1,6                                                     SU23   62
      L=K+2                                                             SU23   63
8002  XMESH2(I,J,K)=RARF(1,L)                                           SU23   64
      GO TO 1000                                                        SU23   65
1003  L=I+1                                                             SU23   66
      N=TEST(Z(L),R(J))                                                 SU23   67
      GO TO (1004,1007,1030,1030,1050),N                                SU23   68
1004  M=PICK2(Z(I),R(J),1)                                              SU23   69
      DO 1006 K=1,8                                                     SU23   70
      ANS=TAB2(M,K,1)+(TAB2(M+1,K,1)-TAB2(M,K,1))*(R(J)-TAB2(M,2,1))/(TASU23   71
```

```
     1B2(M+1,2,1)-TAB2(M,2,1))                                                  SU23  72
      IF (K.NE.1) GO TO 8005                                                    SU23  73
      ANS1=ANS                                                                  SU23  74
      GO TO 1006                                                                SU23  75
8005  CONTINUE                                                                  SU23  76
      IF (K.EQ.2) GO TO 1006                                                    SU23  77
      KX=K-2                                                                    SU23  78
      IF (ABS(R(J)-RADIUS).GT.EPS) GO TO 9024                                   SU23  79
      IF (K.GT.5.OR.K.LT.4) GO TO 9024                                          SU23  80
      ANS=TAB2(M+1,K,1)                                                         SU23  81
9024  CONTINUE                                                                  SU23  82
      XMESH2(I,J,KX)=ANS+(XMESH2(L,J,KX)-ANS)*(Z(I)-ANS1)/(Z(L)-ANS1)           SU23  83
      IF (ABS(R(J)-RADIUS).GT.EPS) GO TO 1006                                   SU23  84
      XMESH2(I,J,1)=0.                                                          SU23  85
      XMESH2(I,J,4)=RHOSTR                                                      SU23  86
      XMESH2(I,J,5)=0.                                                          SU23  87
      XMESH2(I,J,6)=SQRT(BIGAPR/RHOSTR)                                         SU23  88
1006  CONTINUE                                                                  SU23  89
      GO TO 1000                                                                SU23  90
1007  MM=PICK2(Z(I),R(J),2)                                                     SU23  91
      M=PICK2(Z(I),R(J),1)                                                      SU23  92
      DO 1009 K=1,8                                                             SU23  93
      ANS=TAB2(M,K,1)+(TAB2(M+1,K,1)-TAB2(M,K,1))*(R(J)-TAB2(M,2,1))/(TASU23  94
     1B2(M+1,2,1)-TAB2(M,2,1))                                                  SU23  95
      ANSW=TAB2(MM,K,2)+(TAB2(MM+1,K,2)-TAB2(MM,K,2))*(R(J)-TAB2(MM,2,2)SU23  96
     1)/(TAB2(MM+1,2,2)-TAB2(MM,2,1))                                           SU23  97
      IF (K.NE.1) GO TO 1008                                                    SU23  98
      ANS1=ANS                                                                  SU23  99
      ANS2=ANSW                                                                 SU23 100
      GO TO 1009                                                                SU23 101
1008  CONTINUE                                                                  SU23 102
      IF (K.EQ.2) GO TO 1009                                                    SU23 103
      KX=K-2                                                                    SU23 104
      XMESH2(I,J,KX)=ANS+(ANSW-ANS)*(Z(I)-ANS1)/(ANS2-ANS1)                     SU23 105
1009  CONTINUE                                                                  SU23 106
      GO TO 1000                                                                SU23 107
```

```
1010  IF (IRARF.EQ.1) GO TO 1013                                          SU23 108
      M=PICK2(Z(I),R(J),3)                                                SU23 109
      FF=R(J)-RARF2(M+1,2)+(RARF2(M,2)-RARF2(M+1,2))*(Z(I)-RARF2(M+1,1))SU23 110
     1/(RARF2(M,1)-RARF2(M+1,1))                                          SU23 111
      IF (FF.GT.0.) GO TO 1013                                            SU23 112
      DO 1012 K=1,6                                                       SU23 113
      L=K+2                                                               SU23 114
1012  XMESH2(I,J,K)=RARF(1,L)                                             SU23 115
      GO TO 1000                                                          SU23 116
1013  L=I-1                                                               SU23 117
      IF (ABS(Z(I)).LT.EPS) GO TO 1017                                    SU23 118
      N=TEST(Z(L),R(J))                                                   SU23 119
      GO TO (1014,1017,1030,1017,1051),N                                  SU23 120
1014  M=PICK2(Z(I),R(J),2)                                                SU23 121
      DO 1016 K=1,8                                                       SU23 122
      ANS=TAB2(M,K,2)+(TAB2(M+1,K,2)-TAB2(M,K,2))*(R(J)-TAB2(M,2,2))/(TASU23 123
     1B2(M+1,2,2)-TAB2(M,2,2))                                            SU23 124
      IF (K.NE.1) GO TO 1015                                              SU23 125
      ANS1=ANS                                                            SU23 126
      GO TO 1016                                                          SU23 127
1015  CONTINUE                                                            SU23 128
      IF (K.EQ.2) GO TO 1016                                              SU23 129
      KX=K-2                                                              SU23 130
      XMESH2(I,J,KX)=ANS+(XMESH2(L,J,KX)-ANS)*(Z(I)-ANS1)/(Z(L)-ANS1)     SU23 131
1016  CONTINUE                                                            SU23 132
      GO TO 1000                                                          SU23 133
1017  L=J-1                                                               SU23 134
      N=TEST(Z(I),R(L))                                                   SU23 135
      GO TO (1024,1030,1030,1030,1051),N                                  SU23 136
1024  M=PICK2(Z(I),R(J),2)                                                SU23 137
      DO 1026 K=2,8                                                       SU23 138
      ANS=TAB2(M,K,2)+(TAB2(M+1,K,2)-TAB2(M,K,2))*(Z(I)-TAB2(M,1,2))/(TASU23 139
     1B2(M+1,1,2)-TAB2(M,1,2))                                            SU23 140
      IF (K.NE.2) GO TO 1025                                              SU23 141
      ANS1=ANS                                                            SU23 142
      GO TO 1026                                                          SU23 143
```

```
1025  CONTINUE                                                                SU23 144
      KX=K-2                                                                  SU23 145
      IF (ABS(Z(I)).GT.EPS) GO TO 9125                                        SU23 146
      IF (K.GT.5.OR.K.LT.4) GO TO 9125                                        SU23 147
      ANS=TAB2(M+1,K,2)                                                       SU23 148
9125  CONTINUE                                                                SU23 149
      XMESH2(I,J,KX)=ANS+(XMESH2(I,L,KX)-ANS)*(R(J)-ANS1)/(R(L)-ANS1)         SU23 150
      IF (ABS(Z(I)).GT.EPS) GO TO 1026                                        SU23 151
      XMESH2(I,J,1)=0.                                                        SU23 152
      XMESH2(I,J,4)=RHOSTR                                                    SU23 153
      XMESH2(I,J,5)=0.                                                        SU23 154
      XMESH2(I,J,6)=SQRT(BIGAPR/RHOSTR)                                       SU23 155
1026  CONTINUE                                                                SU23 156
      GO TO 1000                                                              SU23 157
1100  M=PICK2(Z(I),R(J),3)                                                    SU23 158
      ANS1=RARF2(M+1,2)+(RARF2(M,2)-RARF2(M+1,2))*(Z(I)-RARF2(M+1,1))/(RSU23 159
     1ARF2(M,1)-RARF2(M+1,1))                                                 SU23 160
      L=J+1                                                                   SU23 161
      M=TEST(Z(I),R(L))                                                       SU23 162
      GO TO (1101,1030,1051,1030),M                                           SU23 163
1101  CONTINUE                                                                SU23 164
      DO 1106 K=1,6                                                           SU23 165
      LL=K+2                                                                  SU23 166
      XMESH2(I,J,K)=RARF(1,LL)+(XMESH2(I,L,K)-RARF(1,LL))*(R(J)-ANS1)/(RSU23 167
     1(L)-ANS1)                                                               SU23 168
1106  CONTINUE                                                                SU23 169
      GO TO 1000                                                              SU23 170
1050  JJ1=J                                                                   SU23 171
      II1=I                                                                   SU23 172
      KOD1=1                                                                  SU23 173
      GO TO 1000                                                              SU23 174
1051  JJ2=J                                                                   SU23 175
      II2=I                                                                   SU23 176
      KOD2=1                                                                  SU23 177
1000  CONTINUE                                                                SU23 178
      IF (KOD1.EQ.0) GO TO 2000                                               SU23 179
```

```
      M=PICK2(Z(III),R(JJ1),1)                                          SU23 180
      DO 2016 K=1,8                                                     SU23 181
      ANS=TAB2(M,K,1)+(TAB2(M+1,K,1)-TAB2(M,K,1))*(R(JJ1)-TAB2(M,2,1))/(SU23 182
     1TAB2(M+1,2,1)-TAB2(M,2,1))                                        SU23 183
      IF (K.NE.1) GO TO 2005                                            SU23 184
      ANS1=ANS                                                          SU23 185
      GO TO 2016                                                        SU23 186
2005  CONTINUE                                                          SU23 187
      I=III                                                             SU23 188
      J=JJ1                                                             SU23 189
      L=III+1                                                           SU23 190
      IF (K.EQ.2) GO TO 2016                                            SU23 191
      KX=K-2                                                            SU23 192
      XMESH2(I,J,KX)=ANS+(XMESH2(L,J,KX)-ANS)*(Z(I)-ANS1)/(Z(L)-ANS1)   SU23 193
2016  CONTINUE                                                          SU23 194
2000  IF (KOD2.EQ.0) GO TO 3000                                         SU23 195
      M=PICK2(Z(II2),R(JJ2),2)                                          SU23 196
      DO 3016 K=1,8                                                     SU23 197
      ANS=TAB2(M,K,2)+(TAB2(M+1,K,2)-TAB2(M,K,2))*(R(JJ2)-TAB2(M,2,2))/(SU23 198
     1TAB2(M+1,2,2)-TAB2(M,2,2))                                        SU23 199
      IF (K.NE.1) GO TO 3015                                            SU23 200
      ANS1=ANS                                                          SU23 201
      GO TO 3016                                                        SU23 202
3015  CONTINUE                                                          SU23 203
      I=II2                                                             SU23 204
      J=JJ2                                                             SU23 205
      L=II2-1                                                           SU23 206
      IF (K.EQ.2) GO TO 3016                                            SU23 207
      KX=K-2                                                            SU23 208
      XMESH2(I,J,KX)=ANS+(XMESH2(L,J,KX)-ANS)*(Z(I)-ANS1)/(Z(L)-ANS1)   SU23 209
3016  CONTINUE                                                          SU23 210
3000  CONTINUE                                                          SU23 211
      GO TO 3017                                                        SU23 212
1030  WRITE (3,1040) I,J                                                SU23 213
1040  FORMAT (27H0 TIME STEP TOO LARGE AT I=,I4,2HJ=,I4,///)            SU23 214
      CALL EXIT                                                         SU23 215
```

```
3017  CONTINUE                                                                SU23  216
C                                                                             SU23  217
C                                                                             SU23  218
      RETURN                                                                  SU23  219
      END                                                                     SU23  220
      SUBROUTINE EXIT                                                         SU24    1
C                                                                             SU24    2
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPSSU24    3
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LENSU24    4
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS         SU24    5
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2SU24    6
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(SU24    7
     115,4),RPART(15,2)                                                       SU24    8
C                                                                             SU24    9
C                                                                             SU24   10
      COMMON Z0,R0,P0,U0,V0,L0,M0,RHO0,E0,A0,UBAR0,VBAR0                      SU24   11
C                                                                             SU24   12
      COMMON NP,NT,NR,NI,NDEL,ISUB                                            SU24   13
C                                                                             SU24   14
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                         SU24   15
      COMMON DIRCOS                                                           SU24   16
      COMMON TIME                                                             SU24   17
      COMMON IRARF                                                            SU24   18
      COMMON KSTOP                                                            SU24   19
      COMMON TPSI                                                             SU24   20
      COMMON KKK                                                              SU24   21
      REAL L0,M0,LENGTH,MU,K0                                                 SU24   22
      KSTOP=1                                                                 SU24   23
      STOP                                                                    SU24   24
      END                                                                     SU24   25
      SUBROUTINE FGOFI(ZX,RX,SS,QQ)                                           SU25    1
C                                                                             SU25    2
C         COMPUTES SI,QI FOR INTERIOR REGION                                  SU25    3
C         ITERATION FOR ZI,RI                                                 SU25    4
C                                                                             SU25    5
      COMMON CASEID(14),ITS1,ITS2,ITS3,ITS4,ITI1,ITI2,ITI3,ITI4,EPS1,EPSSU25    6
```

```
     12,EPS3,EPS4,EPS5,EPS6,EPI1,EPI2,EPI3,EPI4,EPI5,EPI6,EPI7,VP,AR,LENSU25    7
     1GTH,APR,BPR,BIGAPR,BIGBPR,ESTAR,ALPHA,BETA,RHOSTR,EPRS,RHOS          SU25    8
      COMMON XMESH(20,20,6),XMESH2(20,20,6),Z(20),R(20),SURF(15,8),SURF2SU25    9
     1(15,8),TAB(15,14,2),TAB2(15,14,2),SPART(15,2,2),RARF(15,11),RARF2(SU25   10
     115,4),RPART(15,2)                                                  SU25   11
C                                                                        SU25   12
C                                                                        SU25   13
      COMMON ZO,RO,PO,UO,VO,LO,MO,RHOO,EO,AO,UBARO,VBARO                 SU25   14
C                                                                        SU25   15
      COMMON NP,NT,NR,NI,NDEL,ISUB                                       SU25   16
C                                                                        SU25   17
      COMMON ZMIN,ZMAX,RMIN,RMAX,RADIUS,GZ,GR,DELTA,H                    SU25   18
      COMMON DIRCOS                                                      SU25   19
      COMMON TIME                                                        SU25   20
      COMMON IRARF                                                       SU25   21
      COMMON KSTOP                                                       SU25   22
      COMMON TPSI                                                        SU25   23
      COMMON KKK                                                         SU25   24
      REAL LO,MO,LENGTH,MU,KO                                            SU25   25
C                                                                        SU25   26
      DIMENSION ANS(6),PSI(4),SPSI(11),CPSI(11)                          SU25   27
C                                                                        SU25   28
C                                                                        SU25   29
C                                                                        SU25   30
C                                                                        SU25   31
C                                                                        SU25   32
      CALL DBLTRP(ZX,RX,ANS)                                             SU25   33
      UI=ANS(2)                                                          SU25   34
      VI=ANS(3)                                                          SU25   35
      AI=ANS(6)                                                          SU25   36
      SS=ZX-ZO+H*(VI+AI*SIN(TPSI))                                       SU25   37
      QQ=RX-RO+H*(UI+AI*COS(TPSI))                                       SU25   38
      RETURN                                                             SU25   39
      END                                                                SU25   40
```

III.1. "Elementary Solutions of Coupled Model Equations in the Kinetic Tehory of Gases"

III.2. "Tricritical Points in Multicomponent Fluid Mixtures"

III.3. "Generalized Scaling Hypothesis in Multicomponent Systems. I. Classification of Critical Points by Order and Scaling at Tricritical Points"

III.4. "Double-Power Scaling Functions Near Tricritical Points"

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AIAA Paper
No. 69-355

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AIAA Hypervelocity Impact Conference

CINCINNATI, OHIO/APRIL 30 – MAY 2, 1969

STRESS WAVES RESULTING FROM HYPERVELOCITY IMPACT[†]

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Abstract

Results from a numerical scheme based on the method of characteristics are presented for the axially symmetric, hypervelocity impact of similar materials. The analysis is restricted to the early stages of the impact of a right circular cylinder on a halfspace. The resulting rarefaction and shock waves produced by the impact are considered as discrete wavefronts which divide the impacted zone into regions. Numerical diffusion is then controlled by requiring that the values of the dependent variables at a given point in the impacted zone to depend on only the calculated values at earlier times at points in the same region as the point in question. The numerical results give accurate representations of the stress wave profiles (i.e., rarefaction and shock waves) which should be useful as inputs for a later stage elastoplastic analysis and/or spallation analysis. The effects of "numerical diffusion" on the calculated pressure and flow fields when the rarefaction wave is not considered as discrete is investigated and the "diffused" results are compared with the more exact analysis.

Nomenclature

a = isentropic speed of sound
e = specific internal energy
e_s = sublimation energy
h = time step
P = pressure
r = radial coordinate
t = time
U = radial velocity
V = axial velocity
z = axial coordinate
η = ρ/ρ^*
θ = parameter defining a bicharacteristic, $(0 \leq \theta \leq 2\pi)$
μ = $\eta - 1$
ρ = mass density
ρ^* = ambient density
ϕ = function of r, z, and t

Introduction

Most of the existing numerical codes[1-3] for hypervelocity impact calculations are based on procedures which allow some smearing of the resulting rarefaction and shock waves. The present analysis of the initial hydrodynamic phase of an axially symmetric, hypervelocity impact is based on the method of characteristics while treating the rarefaction and shock waves as discrete discontinuities. The definitiveness of the resulting flow field obtained from such an analysis should be useful as inputs to a subsequent elastoplastic analysis for studying later stage cratering effects and/or spallation calculations which require rather accurate representations of both the peak pressure and the incident shock wave profile at reflection. The results based on a numerical code using the method of characteristics for the impact of a right circular aluminum cylinder on an aluminum halfspace are presented. The impact configuration is depicted schematically in Fig. 1.

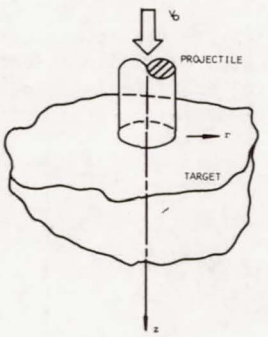

Figure 1 Impact configuration

In order to give some insight into the mechanism of "numerical diffusion", an analysis using the method of characteristics by not considering the rarefaction as discrete is also considered. In

[†]The authors are grateful to Mr. P. Townsend of Northeastern University for his assistance in adapting the computer program to the Northeastern University Computer, the running of the programs, and the preparation of the figures. The research of one of us (T.S.C.) was partially supported by the National Aeronautics and Space Administration under Contract No. NGR 34-002-084, and the Advanced Research Projects of the Department of Defense while monitored by the Office of Naval Research under Contract No. N00014-68-A-0187.

doing so, the speed of the release wave from the free surface of the projectile and the free surface of the target is not controlled. Results from such an approximate analysis are compared with the discrete rarefaction case and the limitations of the "diffused" solution are delineated.

Characteristic Surfaces

As described in references 4 and 5, during the initial stages of a hypervelocity impact, the material in the impacted zone may be assumed to obey the inviscid fluid dynamic equations. For axially symmetric impact, these equations may be expressed as follows:

Conservation of Mass

$$\frac{D\rho}{Dt} + \rho \left(\frac{\partial U}{\partial r} + \frac{\partial V}{\partial z} + \frac{U}{r}\right) = 0 , \qquad (1)$$

Conservation of Momentum

$$\rho \frac{DU}{Dt} + \frac{\partial P}{\partial r} = 0 , \qquad (2)$$

$$\rho \frac{DV}{Dt} + \frac{\partial P}{\partial z} = 0 , \qquad (3)$$

Conservation of Entropy along a Particle Path

$$\frac{DP}{Dt} - a^2 \frac{D\rho}{Dt} = 0 , \qquad (4)$$

where,

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + U \frac{\partial}{\partial r} + V \frac{\partial}{\partial z} , \qquad (5)$$

and P is the pressure, ρ is the density, a is the isentropic speed of sound, t is the time, and (r,z) and (U,V) are the radial and axial coordinates and velocities, respectively.

Equations (1) to (4) form a system of nonlinear, hyperbolic, partial differential equations. The corresponding characteristic equation for this system of equations is:

$$\begin{vmatrix} 0 & \frac{D\phi}{Dt} & \rho\frac{\partial\phi}{\partial r} & \rho\frac{\partial\phi}{\partial z} \\ \frac{\partial\phi}{\partial r} & 0 & \rho\frac{D\phi}{Dt} & 0 \\ \frac{\partial\phi}{\partial z} & 0 & 0 & \rho\frac{D\phi}{Dt} \\ \frac{D\phi}{Dt} & -a^2\frac{D\phi}{Dt} & 0 & 0 \end{vmatrix} = 0 \qquad (6)$$

where $\phi = \phi(r,z,t)$, and surfaces of ϕ = constants are the characteristic surfaces.

The solutions of Eq. (6) are given by

$$\left(\frac{D\phi}{Dt}\right)^2 - a^2 \left[\left(\frac{\partial\phi}{\partial r}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right] = 0 , \qquad (7)$$

and

$$\frac{D\phi}{Dt} = 0 . \qquad (8)$$

The characteristic surfaces described by Eqs. (7) and (8) correspond to two distinct types of characteristic propagations. Corresponding to Eq. (7), the characteristic propagation is described by the differential equations of the bicharacteristics

$$\frac{dr}{dt} = U + a\cos\theta , \qquad (9)$$

$$\frac{dz}{dt} = V + a\sin\theta , \qquad (10)$$

where, $0 \le \theta \le 2\pi$. These bicharacteristics describe the propagation of pressure, density, or velocity disturbances. Corresponding to Eq. (8), the characteristic propagation is described by

$$\frac{dr}{dt} = U , \qquad (11)$$

$$\frac{dz}{dt} = V , \qquad (12)$$

which are the particle path equations and describe the propagation of entropy disturbances.

The set of bicharacteristics from a point, with $0 \le \theta \le 2\pi$, forms the conoid of dependence for the point which is at its vertex. The interior of the conoid contains all the points whose values of U,V,P,ρ, etc. may influence the corresponding values at the point at the vertex. The intersection of this conoid and a particular time plane forms the domain of dependence of the point at the vertex of the conoid and contains all the points whose values of U,V,P,ρ, etc. influence the corresponding values at the point at the vertex. A schematic representation of these ideas are given in Fig. 2.

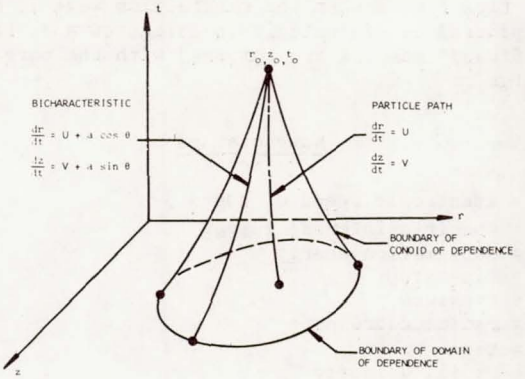

Figure 2 Conoid and domain of dependence

Compatibility Conditions

It will not be difficult to demonstrate that along a bicharacteristic, U,V, and P must satisfy the following differential relation in a time interval dt:

$$dP + \rho a\cos\theta dU + \rho a\sin\theta dV$$

$$= -\rho a^2 dt \left[\left(\frac{\partial U}{\partial r}\right)\sin^2\theta - \left(\frac{\partial U}{\partial z} + \frac{\partial V}{\partial r}\right)\sin\theta\cos\theta + \left(\frac{\partial V}{\partial z}\right)\cos^2\theta + \frac{U}{r}\right] . \qquad (13)$$

Similarly, along the particle path, the following differential expression must be satisfied:

$$\rho^2 de = pd\rho \ , \qquad (14)$$

where e is the specific internal energy. Equations (13) and (14) are called the bicharacteristic and particle path compatibility conditions, respectively.

Equation (14) is essentially the first law of thermodynamics expressed along the particle path for an isentropic process. This choice of a particle path compatibility equation was made in order to introduce the internal energy variable. The P,ρ,e variables are related by an equation of state of the form,

$$P = P(\rho, e) \ . \qquad (15)$$

The particular form of the equation of state used in this analysis is that of Tillotson.[6] For aluminum, it can be expressed as follows:

For $\rho > \rho_*$, with $e > 0$ or $e \leq e_s$,

$$P = [0.5 + \frac{1.63}{1+e/(e_s\eta^2)}]e\rho + 0.752\mu + 0.65\mu^2 \ , \qquad (16)$$

and for $\rho < \rho_*$, with $e > e_s$,

$$P = 0.5e\rho + [\frac{1.63e\rho}{1+e/(e_s\eta^2)} + 0.752\mu e^{+5(1-1/\eta)}]e^{-5(1-1/\eta)^2} \ , \qquad (17)$$

where,

$$\eta = \rho/\rho_* \ , \quad \mu = \eta - 1 \ , \qquad (18)$$

$\rho_* = 2.70\ gm/cm^3$ is the ambient density for aluminum, and $e_s = 0.03$ megabar cm^3/gm is the sublimation energy for aluminum. In Eqs. (16) and (17), P is expressed in megabars, ρ in gm/cm^3, and e in megabars cm^3/gm.

Numerical Procedure

The differential expressions, Eqs. (9)-(14), are cast in finite difference forms accurate to the order of h^2, where h is the time step. For example, the bicharacteristic compatibility equation for a particular value of θ is

$$P_o - P_i + \rho_i a_i [\cos\theta_i(U_o - U_i) + \sin\theta_i(V_o - V_i)] = -\rho_i a_i^2 h [(\frac{\partial U}{\partial r})_i \sin^2\theta_i - (\frac{\partial U}{\partial z} + \frac{\partial V}{\partial r})_i \sin\theta_i\cos\theta_i + (\frac{\partial V}{\partial z})_i \sin^2\theta_i + \frac{U_i}{r_i}] + O(h^2) \ , \qquad (19)$$

where the subscript "o" refers to $t = t_o$ and the point where the values of U,V,P,ρ, etc. are being calculated and the subscript "i" refers to the corresponding values at the intersection of the bicharacteristic θ_i with an (t_o-h) - time plane at which all data has been previously calculated. In spite of the fact that the error over each time step is of order h^2, the accumulated errors reduce the aggregate solution to the order of h.

The unknowns in Eq. (19) are P_o, U_o, V_o and therefore unless one (or more) of these values is specified, three bicharacteristics are required for each calculation. After P_o, U_o, V_o are determined, the values ρ_o and e_o are obtained with the aid of the particle path compatibility condition and the equation of state. This procedure is applicable at points not on the boundary of the impacted zone, (i.e., on the shocks, rarefaction, free surfaces, or axis of symmetry). At points on the boundary, auxiliary relations are available and consequently, fewer independent bicharacteristic conditions are required for each calculation. A scheme is also used to insure that finite difference relations are not written across discontinuities and "numerical diffusion" across these discontinuities is minimized. The details of the various calculational procedures are given in references 4 and 5.

Numerical Stability

The problem of constructing a stable numerical procedure for the present analysis is resolved by following certain qualitative guidelines. The basic idea is suggested by the stability criterion of Courant, Friedrichs, and Lewy, (or the C.F.L. Condition). According to this condition, the domain of dependence of the difference equations must encompass the domain of dependence of the original hyperbolic differential equations for numerical stability. The following is a brief account of the approach used in this analysis to ascertain numerical stability.

Consider the domain of dependenee of the partial differential equations in the (t_o-h) - time plane as illustrated in Fig. 3(a). If the conoid of dependence is approximated by the three bicharacteristics passing through the triangular points (1,2,3), the domain of dependence is described by the triangle of dashed lines connecting the three points. Since, for the purpose of attaining numerical convergence, the points (1,2,3) must be relatively close to the circular region which is the domain of dependence of the differential equations, instabilities cannot be avoided. To insure numerical stability, the values at points (1,2,3) are obtained by interpolation using the values at the twelve grid points (circular dots) which surround the three original points as shown in Fig. 3(b). In doing so, the domain of depend-

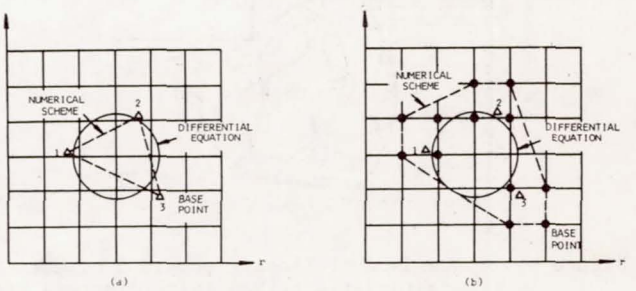

Figure 3 Domains of dependence: (a) without interpolation, (b) with interpolation

ence of the difference equations is increased to the convex hull of dashed lines indicated in Fig. 3(b). This convex hull should encompass the circular domain of the differential equations. Such an interpolation scheme seems to be sufficient to insure the required numerical stability for the present analysis.

Results and Discussion

The example chosen in this paper to illustrate the results of the characteristic method is the axially symmetric impact of an infinitely long cylindrical aluminum projectile with a diameter d_o equal to 2.5 cm on an aluminum halfspace. The initial velocity V_o of the projectile is taken to be 7.6 km/sec and the halfspace is assumed to be at rest. The results of the characteristic method are compared with the results obtained from the well-known "OIL Code" of reference 1. Results from the discrete rarefaction case is then compared with those for a case where the "numerical diffusion" of the rarefaction wave is allowed.

Figure 4 shows the pressure distributions for both the "OIL" solution of reference 1 and the method of characteristics with discrete rarefaction at a time of t = 0.74 μsec after impact when the rarefaction has not yet propagated near the axis of symmetry. The pressure distributions are compared by mapping the isobars for P = 1.08, 0.8, 0.5, and 0.2 megabars (MB). The results of the method of characteristics are shown as solid lines and the "OIL" distributions as dashed lines. The locations of the two shocks and the rarefaction (P = 1.08 MB) are quite clearly defined for the characteristic method. One difference immediately apparent between the two sets of results is that the isobars for the "OIL" approach begin and end on the axis of symmetry, whereas the isobars for the characteristic method begin and end on the projectile and target shocks. The difference, of course, comes from the smearing of the shock fronts in the "OIL" solution. The "OIL" approach predicts additionally two regions of pressure higher than the one-dimensional region which is bounded by the two shocks and the rarefaction. The reduction in pressure in the rarefied region compares quite favorably for both methods. Major differences among the isobars of P = 0.5 and 0.2 MB obtained from the two methods are noted in the region near the original free surface where the pressure has been controlled by the characteristic method.

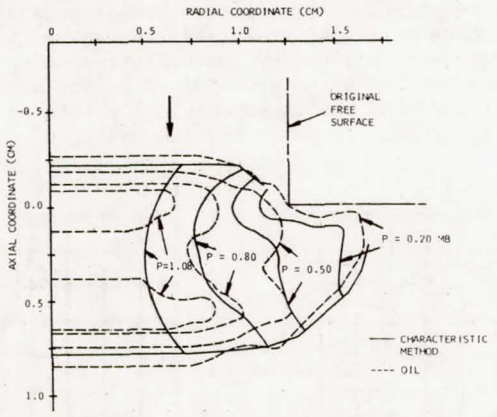

Figure 4 Comparison of isobars obtained by the characteristic method (with discrete rarefaction) and the "OIL" results. V_o = 7.6 km/sec, t = 0.74 μsec, d_o = 2.5 cm

Figure 5 shows the pressure distribution along the axis of symmetry at a time of t = 1.28 μsec after impact just prior to the actual reflection of the rarefaction wave from the axis of symmetry. The dashed curve shows the correct uniform pressure distribution based on the discrete rarefaction solution. The solid line shows the pressure distribution predicted by the numerical method which allows "diffusion" across the rarefaction wave. In the latter case, the pressure along the axis is underestimated. The lowest value is approximately 60% of the actual value. This drop in pressure is due to the fact that the effective speed of the rarefaction for the "diffused" case is equal to the actual rarefaction speed plus the mesh speed (i.e., the rarefaction front travels one additional grid spacing in each time cycle). The fact that the pressures are in error by 40% could be serious if the target shock wave were to be reflected from a free surface at the back of a target with a finite thickness or if these results were to be used as inputs for an elastoplastic analysis for later stage cratering and subsequent spallation calculations.

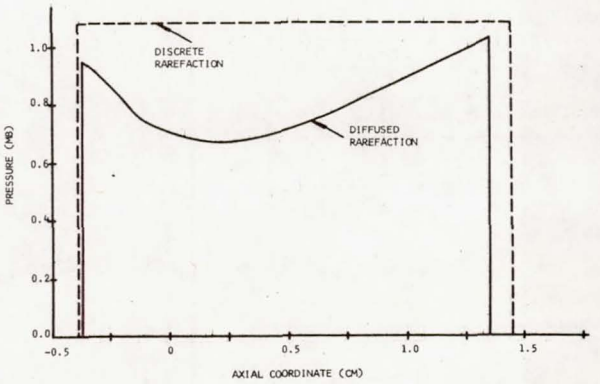

Figure 5 Pressure distribution along the axis of symmetry. V_o = 7.6 km/sec, t = 1.28 μsec, d_o = 2.5 cm

Figure 6 shows the pressure distribution along the axis of symmetry at a later time (t = 1.55 μsec) after the rarefaction has been re-

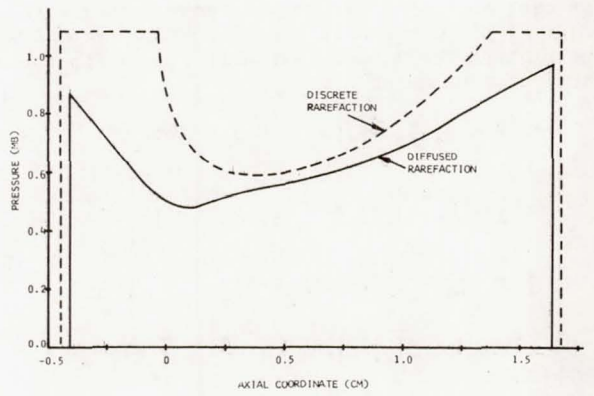

Figure 6 Pressure distribution along the axis of symmetry. V_o = 7.6 km/sec, t = 1.28 μsec, d_o = 2.5 cm

flected. Again the dashed curve is that of the discrete rarefaction solution and the solid curve is the "diffused" solution. At or near this time and for subsequent times, the results of the two solutions are very close to each other and the discrepancies among the results diminish with the increase of time.

Figure 7 maps the isobars for the entire flow field for both the discrete rarefaction solution (dashed curves) and the "diffused" solution (solid curves) at t = 1.28 μsec after impact. It is noted that in regions close to the axis of symmetry and the rarefaction, there are large discrepancies among the numerically calculated pressures using the two different methods. In addition, due to the premature decay of the projectile shock near the projectile free surface, the calculated pressures near the target shock close to the free surface using the "diffused" scheme are much higher than those for the discrete rarefaction solution. This effect also causes a slightly higher target shock velocity near the free surface. In other regions far away from the free surface and the rarefaction front, the calculated pressures from the two methods are in good agreement.

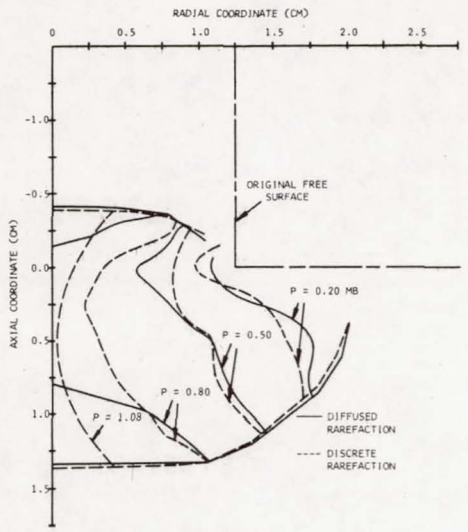

Figure 7 Pressure field. V_o = 7.6 km/sec t = 1.28 μsec, d_o = 2.5 cm

Figure 8 shows a comparison of the isobars in the impacted zone at t = 1.55 μsec after impact. At this time, the isobars are in close agreement throughout the flow field. Nevertheless, some of the effects of the "diffused" rarefaction can still be observed. For instance, there is a slightly higher pressure region predicted by the "diffused" results near the target shock close to the axis of symmetry.

Figure 9 compares the velocity vectors for the two methods of calculations at t = 1.50 μsec after impact. For all practical purposes, the results are nearly identical. It is apparent that the effect of "numerical diffusion" is more important in the pressure calculations.

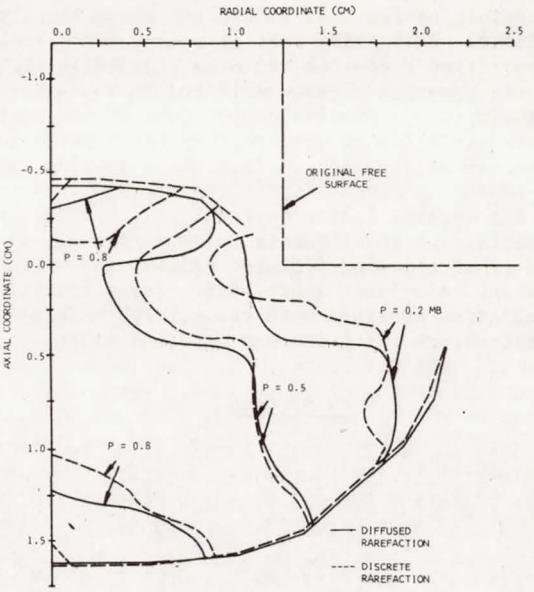

Figure 8 Pressure fields. V_o = 7.6 km/sec, t = 1.55 μsec, d_o = 2.5 cm

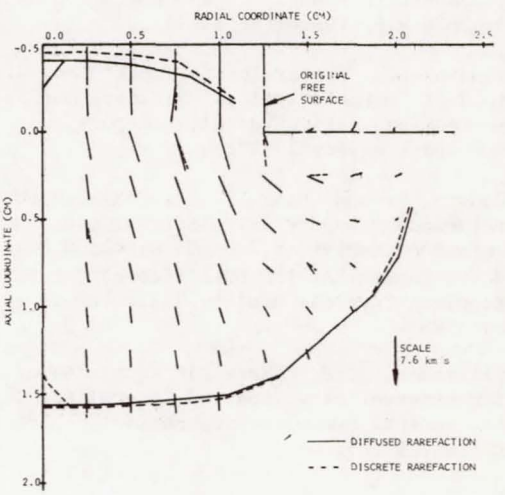

Figure 9 Velocity vectors. V_o = 7.6 km/sec, t = 1.50 μsec, d_o = 2.5 cm

Concluding Remarks

It may be surmised from the calculated results using the method of characteristics that, if the rarefaction is not considered as a discrete discontinuity, the pressures for the initial stages of an axially symmetric hypervelocity impact may be quite different from their actual values before the rarefaction is reflected from the axis of symmetry. This is particularly true for the pressure distribution at the axis of symmetry. After the reflec-

tion of the rarefaction wave, however, the effect of numerical "diffusion" introduced by not treating the rarefaction front as a discrete discontinuity diminishes. This means that if the pressure fields and shock fronts are desired at a time after the rarefaction wave has been reflected from the axis of symmetry, an approximate procedure of the method of characteristics by not treating the rarefaction as discrete may be used. Since the computing time required for hypervelocity impact calculations using the method of characteristics with discrete rarefaction and shock fronts is very large, this procedure of allowing diffused rarefaction can be useful for relatively long impact time calculations or spallation analyses with relatively thick target plates (compared with the projectile diameter).

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75-04939

Reprinted from

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PERGAMON PRESS
OXFORD NEW YORK LONDON PARIS

Int. J. Engng Sci., 1972, Vol. 10, pp. 353–367. Pergamon Press. Printed in Great Britain

NONLINEAR WAVES IN A RATE-SENSITIVE, ELASTOPLASTIC MATERIAL

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(Communicated by A. C. ERINGEN)

Abstract—Two classes of closed form solutions of one-dimensional, nonlinear waves in a rate-sensitive, elastoplastic material are reported. One class of these solutions is self-similar and the other class consists of constant speed propagations. Applications of these solutions to unsteady motions behind propagating discontinuities are also considered.

1. INTRODUCTION

THE PURPOSE of this paper is to discuss two interesting classes of closed form solutions of one-dimensional, unsteady motion of a rate-sensitive, elastoplastic material. One class of these solutions is self-similar and is deduced from the invariant theorems of continuous groups of transformations. This class of unsteady motion is governed by a single, first-order, nonlinear, ordinary differential equation of the Riccati type and closed form solutions in terms of elementary functions are obtained under special circumstances. If the material in consideration possesses the additional property of instantaneous linear elasticity [1] under 'high rate' of straining, it may be demonstrated that one of these self-similar solutions can be used to describe the dispersed nonlinear wave motion behind a propagating shockfront into an initially quiescent region.

The second class of solutions is obtained by searching for one-dimensional wave motions with constant speeds of propagation. These solutions are expressible as simple quadratures and closed form expressions can be obtained for specific constitutive relations. Such solutions represent non-characteristic propagations, i.e. they are not propagations of weak discontinuities of arbitrary wave forms. It may be demonstrated, using the Poincaré–Bendixon theorem, that these solutions, in general, are not periodic. Assuming a sub-elastic, constant-speed, propagating discontinuity preceded by an elastic precurser with an unloading, relaxation zone, or a constant stress region, the nonlinear wave solution with a constant propagation speed equal to that of the discontinuity can be used to describe the 'unsteady'‡ motion behind the discontinuity.

One-dimensional rectilinear motion, in the strict sense, involves not just one spatial coordinate but also only one component of stress, strain, and particle velocity. For such a type of motion, only a one-dimensional stress–strain or constitutive relation is required. Various rate-sensitive, constitutive equations have been proposed in the

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‡Such an 'unsteady' motion, of course, becomes essentially steady for a moving observer following the propagating discontinuity.

literature and a comprehensive review of this subject can be found in Cristescu [2]. The solutions described in this paper are obtained based on a model first proposed by Sokolovskii [3, 4]. This model cannot, in general, be used to describe the structure or generation of shock waves [5, 6]. In applying the solutions given in this paper to the unsteady motions behind propagating shock layers or relaxation zones, additional material properties may have to be assumed within these shock or relaxation regions.

2. MATHEMATICAL FORMULATION

One-dimensional motion may be described by a scalar deformation field,

$$\bar{x} = \bar{x}(\bar{X}, \bar{t}), \tag{2.1}$$

where $\bar{x}$ is the instantaneous position coordinate at time $\bar{t}$ of a generic particle whose position coordinate at $\bar{t} = 0$ was $\bar{X}$. The Lagrangian equation of motion and kinematic compatibility condition for rectilinear, one-dimensional motion are†

$$\rho \partial u/\partial \bar{t} = \partial \bar{\sigma}/\partial \bar{X}, \tag{2.2}$$

$$\partial \epsilon/\partial \bar{t} = \partial u/\partial \bar{X}, \tag{2.3}$$

where $\bar{\sigma}$ is the longitudinal stress, and

$$u = \partial \bar{x}/\partial \bar{t}, \tag{2.4}$$

$$\epsilon = \partial \bar{x}/\partial \bar{X} - 1. \tag{2.5}$$

are the particle velocity and linear Lagrangian strain, respectively. The material is assumed to be initially unstressed and unstrained with a constant density ρ.

In this analysis, the material under consideration will be assumed to follow the special constitutive relation for a rate-sensitive, elastoplastic material generalized from a model suggested by Sokolovskii [3, 4],

$$\partial \epsilon/\partial \bar{t} = \partial u/\partial \bar{X} = E^{-1} \partial \bar{\sigma}/\partial \bar{t} + \gamma f(\bar{\sigma}/\sigma_0 - 1) 1_+(\bar{\sigma}/\sigma_0 - 1), \tag{2.6}$$

where $f(\cdot)$ is a dimensionless C^1 function with $f(\eta) > 0$ for $\eta > 0$, $1_+(\cdot)$ is the Heaviside function, E is the modulus of elasticity which is assumed to be a constant, σ_0 is the static yield stress, and γ is a material constant. Thus, the material is assumed to have an elastic range with a constant modulus. If the strain rate is held constant, Equation (2.6) may be integrated by simple quadrature‡. Typical stress–strain curves for $\partial \epsilon/\partial \bar{t} =$ constant are displayed in Fig. 1. It is seen that the dynamic yield stress is rate-sensitive and there are no strain-hardening effects. Equation (2.6) includes the well-known models suggested by Cowper and Symonds [8], and Perzyna [9], as special cases. It is a special form of a more general constitutive equation suggested by Malvern [10].

Equations (2.2) and (2.6) are the basic equations describing the functions, $u(\bar{X}, \bar{t})$, $\bar{\sigma}(\bar{X}, \bar{t})$, [and $\epsilon(\bar{X}, \bar{t})$], characterizing the one-dimensional motions to be considered in this paper. These equations may be combined into one single, second-order, nonlinear,

†See e.g. Courant and Friedrichs [7].

‡The authors are grateful to Professor A. C. Eringen for suggesting this.

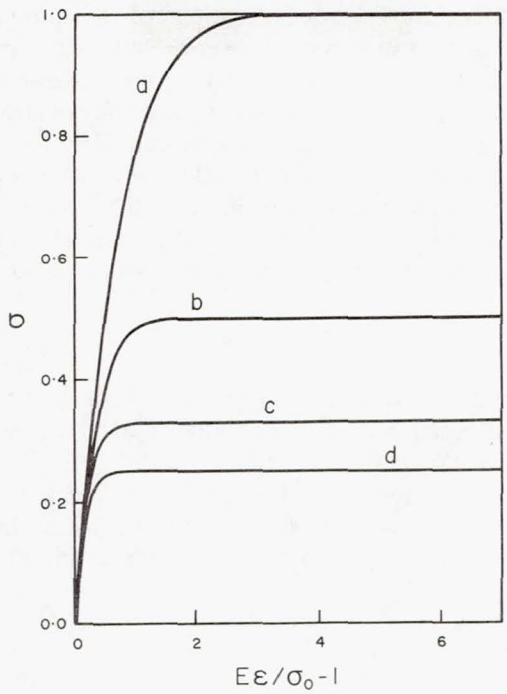


Fig. 1. Typical stress–strain curves for constant strain rates. (a) $\sqrt{\gamma}/k = 1$, (b) $\sqrt{\gamma}/k = 2$, (c) $\sqrt{\gamma}/k = 3$, (d) $\sqrt{\gamma}/k = 4$, $k = \partial\epsilon/\partial t$.

hyperbolic, partial differential equation of the evolution type in dimensionless form as follows:

$$\beta(\partial^2\sigma/\partial x^2 - \partial^2\sigma/\partial t^2) = [1_+(\sigma)\mathrm{d}f(\sigma)/\mathrm{d}\sigma + \delta(\sigma)f(0)]\partial\sigma/\partial t, \tag{2.7}$$

where

$$\sigma(x,t) \equiv \bar{\sigma}/\sigma_0 - 1, \tag{2.8}$$

is the dimensionless overstress,

$$x \equiv \alpha\bar{X}, \tag{2.9}$$

$$t \equiv \alpha c\bar{t}, \tag{2.10}$$

$$c \equiv \sqrt{E/\rho}, \tag{2.11}$$

$$\alpha \equiv \beta\gamma\sqrt{\rho E/\sigma_0}, \tag{2.12}$$

$\delta(\cdot)$ is the Dirac delta functional, and $\beta > 0$ is a dimensionless constant included here in the definition of α for convenience.

Materials described by the constitutive relation given in (2.6) probably cannot support shock layers or explain the generation of shockfronts. If a shock layer is dissipative, then generalized viscoelastic theories and constitutive relations such as those considered by Varley and Rogers[6], Coleman and Gurtin[11], Dunwoody and Dun-

woody [12] and Pipkin [5], or further generalizations of these models, should be used to describe it. For propagation in a rod, the shock layer may be dispersive due to lateral deformation† instead of due to any dissipative mechanism. Such a shock transition may be described in terms of a low frequency, large rate of straining expansion of a three-dimensional deformation field similar to that considered by Parker and Varley [13]. In applying one of the self-similar motions described in this paper to a nonlinear wave motion behind a propagating shockfront, it will be assumed that the rate of straining in the shock layer is high enough to allow the material to exhibit instantaneous linear elasticity‡ [1]. Thus, across such a shock layer, it will be assumed that

$$[\bar{\sigma}] = E[\epsilon], \tag{2.13}$$

where $[\chi]$ denotes the jump in value of χ across the shock layer, and the value of E will be assumed to be a constant and have the same value as the modulus of the elastic range of the constitutive relation given by (2.6).

From the Lagrangian equation of motion (2.2) and the kinematic compatibility condition (2.3), two additional jump conditions relating $[u]$, and $[\epsilon]$, can be deduced formally following a technique suggested by Courant and Friedrichs [7]. The results are:

$$\rho U[u] + [\bar{\sigma}] = 0, \tag{2.14}$$

$$U[\epsilon] + [u] = 0, \tag{2.15}$$

where U is the propagation speed of the shockfront. Equations (2.14) and (2.15) can also be deduced from physical arguments directly. The jump conditions, (2.13)–(2.15), indicate that the speed of propagation of a shock layer of a material exhibiting instantaneous elasticity is

$$|U| = \sqrt{E/\rho}, \tag{2.16}$$

which is, in fact, the same as the elastic speed of propagation of small disturbances.

In applying the constant speed solutions to the 'unsteady' motion behind a propagating discontinuity which moves at a constant sub-elastic speed, it will be assumed that there is an elastic precurser and an unloading, relaxation zone, or a constant stress region, ahead of the discontinuity. The details of the unsteady motion of a relaxation zone ahead of such a discontinuity may be very complicated and will not be considered in this paper.

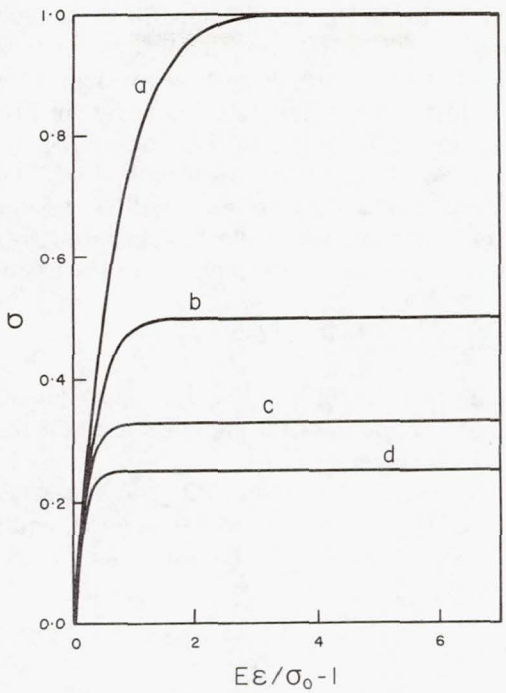
3. A CLASS OF SELF-SIMILAR SOLUTIONS

Cowper and Symonds [8] proposed, in 1957, a power law,

$$f(\sigma) = \sigma^{\delta}, \tag{3.1}$$

†The authors are indebted to Professor E. Varley for a discussion pertaining to this point.

‡The range of rate of straining within which materials exhibit instantaneous elasticity varies from one material to another. There is usually an upper (and lower) cutoff point in rate of straining above (and below) which a material may have to be considered viscoelastic. The authors are indebted to Professor R. S. Rivlin for pointing this out to them.



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$$x \equiv \alpha\bar{X}, \tag{2.9}$$

$$t \equiv \alpha c\bar{t}, \tag{2.10}$$

$$c \equiv \sqrt{E/\rho}, \tag{2.11}$$

$$\alpha \equiv \beta\gamma\sqrt{\rho E/\sigma_0}, \tag{2.12}$$

$\delta(\cdot)$ is the Dirac delta functional, and $\beta > 0$ is a dimensionless constant included here in the definition of α for convenience.

Materials described by the constitutive relation given in (2.6) probably cannot support shock layers or explain the generation of shockfronts. If a shock layer is dissipative, then generalized viscoelastic theories and constitutive relations such as those considered by Varley and Rogers[6], Coleman and Gurtin[11], Dunwoody and Dun-

woody[12] and Pipkin[5], or further generalizations of these models, should be used to describe it. For propagation in a rod, the shock layer may be dispersive due to lateral deformation† instead of due to any dissipative mechanism. Such a shock transition may be described in terms of a low frequency, large rate of straining expansion of a three-dimensional deformation field similar to that considered by Parker and Varley [13]. In applying one of the self-similar motions described in this paper to a nonlinear wave motion behind a propagating shockfront, it will be assumed that the rate of straining in the shock layer is high enough to allow the material to exhibit instantaneous linear elasticity‡[1]. Thus, across such a shock layer, it will be assumed that

$$[\bar{\sigma}] = E[\epsilon], \tag{2.13}$$

where $[\chi]$ denotes the jump in value of χ across the shock layer, and the value of E will be assumed to be a constant and have the same value as the modulus of the elastic range of the constitutive relation given by (2.6).

From the Lagrangian equation of motion (2.2) and the kinematic compatibility condition (2.3), two additional jump conditions relating $[u]$, and $[\epsilon]$, can be deduced formally following a technique suggested by Courant and Friedrichs[7]. The results are:

$$\rho U[u] + [\bar{\sigma}] = 0, \tag{2.14}$$

$$U[\epsilon] + [u] = 0, \tag{2.15}$$

where U is the propagation speed of the shockfront. Equations (2.14) and (2.15) can also be deduced from physical arguments directly. The jump conditions, (2.13)–(2.15), indicate that the speed of propagation of a shock layer of a material exhibiting instantaneous elasticity is

$$|U| = \sqrt{E/\rho}, \tag{2.16}$$

which is, in fact, the same as the elastic speed of propagation of small disturbances.

In applying the constant speed solutions to the 'unsteady' motion behind a propagating discontinuity which moves at a constant sub-elastic speed, it will be assumed that there is an elastic precurser and an unloading, relaxation zone, or a constant stress region, ahead of the discontinuity. The details of the unsteady motion of a relaxation zone ahead of such a discontinuity may be very complicated and will not be considered in this paper.

3. A CLASS OF SELF-SIMILAR SOLUTIONS

Cowper and Symonds[8] proposed, in 1957, a power law,

$$f(\sigma) = \sigma^{\delta}, \tag{3.1}$$

†The authors are indebted to Professor E. Varley for a discussion pertaining to this point.

‡The range of rate of straining within which materials exhibit instantaneous elasticity varies from one material to another. There is usually an upper (and lower) cutoff point in rate of straining above (and below) which a material may have to be considered viscoelastic. The authors are indebted to Professor R. S. Rivlin for pointing this out to them.

where $\delta > 0$ is a dimensionless material constant, to describe the rate-sensitivity of perfectly plastic materials. This law seems to be quite adequate in approximating the dynamic responses of certain metallic alloys[14. 15] under moderately high rates of straining. The material constants, γ and δ, for such a material may be determined by explosive ring-tests as suggested by Perrone[16]. Recent investigators[17–21] have applied this model to impulsively loaded beams, rods, and plates. The class of self-similar solutions described in this paper is based on the constitutive relation (2.6) and the special form of $f(\sigma)$ given by (3.1). Under these constitutive assumptions, equation (2.7) may be expressed as follows:

$$\phi(\sigma_{xx}, \sigma_{tt}, \sigma_t, \sigma; x, t) = 0, \tag{3.2}$$

where

$$\phi \equiv \sigma_{xx} - \sigma_{tt} - \sigma^{\delta-1}\sigma_t 1_+(\sigma), \tag{3.3}$$

and subscripts denote partial differentiation. The constant β which appeared in the definition of α in (2.12) has been replaced by the material constant δ.

Consider a one-parameter continuous group of transformations defined by

$$(X, T, \Sigma) = (bx, b^m t, b^n \sigma), \tag{3.4}$$

$$(\Sigma_{XX}, \Sigma_{TT}, \Sigma_T) = (b^{n-2}\sigma_{xx}, b^{n-2m}\sigma_{tt}, b^{n-m}\sigma_t), \tag{3.5}$$

where b is the parameter, and m,n are constants. It can be shown that for the special case of $m = 1, n = 1/(1-\delta)$,

$$\phi(\sigma_{xx}, \sigma_{tt}, \sigma_t, \sigma; x, t) = b^{(1-2\delta)/(1-\delta)}\phi(\Sigma_{XX}, \Sigma_{TT}, \Sigma_T, \Sigma; X, T), \tag{3.6}$$

where it is assumed that $\delta \neq 1$. For $\delta = 1$, equation (3.2) is linear and the analytical solution has been discussed in detail by Malvern[10]. Thus, ϕ is a constant conformal invariant under the group defined by equations (3.4) and (3.5) with $m = 1$, and $n = 1/(1-\delta)$. According to a theorem proven by Morgan[22], the solution to equation (3.2) may be expressed in terms of a function $F(\xi)$ of an absolute invariant ξ of the transformation group defined by

$$(X, T) = (bx, bt). \tag{3.7}$$

The function $F(\xi)$ is an absolute invariant of the transformation group defined by

$$(X, T, \Sigma) = (bx, bt, b^{1/(1-\delta)}\sigma). \tag{3.8}$$

It will be straightforward to verify that

$$\xi = t/x, \tag{3.9}$$

$$F(\xi) = x^{1/(\delta-1)}\sigma(x, t), \tag{3.10}$$

are absolute invariants of the groups defined by equations (3.7) and (3.8), respectively.

Thus, there exists a class of self-similar solutions to equation (3.2) of the form

$$\sigma = x^{1/(1-\delta)} F(\xi), \tag{3.11}$$

where ξ is given by (3.9).

Substituting equation (3.11) into equation (3.2) and using equation (3.3), a non-linear, second-order ordinary differential equation results. For $\sigma > 0$, this equation is expressible as follows:

$$(\xi^2-1)F''-\{[2\delta/(1-\delta)]\xi+F^{\delta-1}\}F'+[\delta/(1-\delta)^2]F=0, \tag{3.12}$$

where prime denotes differentiation.

For the special case of $\delta = 2$, $\xi > 0$, equation (3.12) is immediately integrable to the following Riccati equation:

$$2(\xi^2-1)F'+4\xi F-F^2=4K, \tag{3.13}$$

where K is an arbitrary constant. This equation may be converted into a linear, second-order, ordinary differential equation by the following transformation:

$$(\xi^2-1)^r V(z) = \mathrm{Exp}\left[-\frac{1}{2}\int^{\xi} \frac{F(\xi')}{\xi'^2-1}\,\mathrm{d}\xi'\right], \tag{3.14}$$

$$2z = \xi + 1. \tag{3.15}$$

The result is:

$$z(1-z)V''+2(1+r)(1-2z)V'-(2r-K)V=0, \tag{3.16}$$

where r satisfies the quadratic equation,

$$4r^2+4r+K=0. \tag{3.17}$$

Equation (3.16) has three regular singular points at $z = 0$, 1, and ∞. The solutions to this equation are expressible in terms of hypergeometric functions. For $\xi > 1$, the appropriate general solution to (3.16) is, in the usual notation,

$$V(z) = Cz^{-2r-3}{}_2F_1[2r+3, 2, 4; 1/z], \tag{3.18}$$

where C is an arbitrary constant. Thus, from equation (3.14), the corresponding expression for $F(\xi)$ is

$$F(\xi) = 6(\xi-1)-4r+2(2r+3)\frac{(\xi-1){}_2F_1[2r+4, 3, 5; 2/(\xi+1)]}{(\xi+1){}_2F_1[2r+3, 2, 4; 2/(\xi+1)]}. \tag{3.19}$$

The expression (3.19) for $F(\xi)$ assumes some particularly simple forms in terms of elementary functions for special values of K. As examples, typical expressions for $F(\xi)$ and $\sigma(x, t)$ for two different values of K are listed below:

K = 0, (i.e. r = 0, or −1)

$$F(\xi) = 8\{2\xi + (\xi^2-1)\ln[(\xi-1)/(\xi+1)] + A_1(\xi^2-1)\}^{-1}. \tag{3.20}$$

$$\sigma(x,t) = 8x\{2xt + (t^2-x^2)\ln[(t-x)/(t+x)] + A_1(t^2-x^2)\}^{-1}. \tag{3.21}$$

K = −3, (i.e. r = 1/2 or −3/2)

$$F(\xi) = 6(1 + A_2\xi + \xi^2)/(A_2 + 3\xi - \xi^2), \tag{3.22}$$

$$\sigma(x,t) = 6(x^2 + A_2xt + t^2)/(A_2x^3 + 3x^2t - t^3). \tag{3.23}$$

In these expressions, A_1 and A_2 are arbitrary constants.

It is interesting to note that the solution given by (3.21) is invariant under the translation defined by $(x', t') = (x+a, t+a)$, where a is an arbitrary constant. This property will be utilized in Section 5 to derive a closed form solution of a self-similar, unsteady, dispersed, nonlinear wave motion behind a constant 'elastic-speed' shockfront propagating into an initially quiescent region.

4. NONLINEAR WAVES WITH CONSTANT SPEEDS

Equation (2.7) is a nonlinear, hyperbolic differential equation of the evolution type. The characteristic speeds related to this equation are given by,

$$\mathrm{D}_{\pm}\bar{X}/\mathrm{D}t = \pm c, \tag{4.1}$$

or,

$$\mathrm{D}_{\pm}x/\mathrm{D}t = \pm 1, \tag{4.2}$$

where $\mathrm{D}_{\pm}(\cdot)/\mathrm{D}\bar{t}$ and $\mathrm{D}_{\pm}(\cdot)/\mathrm{D}t$ denote differentiation along the characteristics.

Due to the presence of the evolution or dissipative term, $[1_+(\sigma)\mathrm{d}f(\sigma)/\mathrm{d}\sigma + \delta(\sigma)f(0)]\cdot\partial\sigma/\partial t$, in equation (2.7), it is expected that, in the plastic range, the material can also support dissipative, dispersive waves in addition to the characteristic propagations of discontinuities given by equation (4.1) or (4.2). To demonstrate the existence of non-characteristic propagations, a class of constant speed solutions to equation (2.7) is considered in this section. This class of solutions is obtained by searching for expressions of the form:

$$\sigma(x,t) = g(s), \tag{4.3}$$

where

$$s \equiv c\bar{t} - x, \tag{4.4}$$

and $\bar{c}$ = constant determines the speed of propagation.

Substituting equation (4.3) into equation (2.7), a nonlinear, second-order, ordinary differential equation results:

$$\beta(1-\bar{c}^2)g'' = \bar{c}[1_+(g)f'(g) + \delta(g)f(0)]g', \tag{4.5}$$

where primes denote differentiation. This equation can be integrated once immediately

to yield,

$$\beta(1-\bar{c}^2)g' = \bar{c}1_+(g)f(g)+A, \tag{4.6}$$

where A is an arbitrary constant. Actually, the fact that equation (4.5) can be integrated once in closed form is not due to the special choice of the constitutive equation, (2.6), since, by assuming solutions of constant speeds of propagation, equation (2.2) can be integrated at once without making any additional assumptions. By comparing the expression (4.6) with the basic equations, (2.2) and (2.6), it is easily demonstrated that $A = 0$. Thus,

$$\beta(1-\bar{c}^2)g' = \bar{c}1_+(g)f(g), \tag{4.7}$$

for constant speeds of propagation.

If $g < 0$, then equation (4.7) becomes

$$(1-\bar{c}^2)g' = 0. \tag{4.8}$$

Except for the trivial case of $g =$ constant, equation (4.8) requires that $\bar{c} = \pm 1$, which of course is the elastic speed of propagation. For $g > 0$, equation (4.7) requires that

$$\text{(i)}\ \ \bar{c}^2 < 1, \text{for}\ (g'/\bar{c}) > 0, \tag{4.9}$$

$$\text{(ii)}\ \ \bar{c}^2 > 1, \text{for}\ (g'/\bar{c}) < 0. \tag{4.10}$$

Thus, for the physically more meaningful case $\bar{c}^2 < 1$, the overstress may increase or decrease with s depending on whether $\bar{c} > 0$ or < 0. Equations (4.9) and (4.10) also indicate that if solutions for $g > 0$ exist, the waves represented by these solutions are not characteristic propagations.†

On setting $h = g'$, equation (4.5) for $g > 0$ may be written as

$$\frac{h'}{g'} = \frac{\bar{c}}{\beta(1-\bar{c}^2)} f'(g). \tag{4.11}$$

Since $f(g)$ is a C^1 function, equation (4.11) does not have any singular points. Thus, according to the Poincaré–Bendixon theorem, it may be concluded that equation (4.11), in general, does not possess periodic solutions.

For $g > 0$, equation (4.7) may be integrated, in general, by quadrature as follows:

$$s = [\beta(1-\bar{c}^2)/\bar{c}] \int^g f^{-1}(\zeta)\, \mathrm{d}\zeta + C', \tag{4.12}$$

where C' is an arbitrary constant.

Perzyna[9] in 1963, suggested two interesting expressions for $f(\zeta)$. In slightly generalized forms, these expressions are given as follows:

†In fact, these waves are similar to the so-called solitary waves which are constant-speed, nonlinear propagations of special wave forms. The authors are indebted to Professor G. S. S. Ludford for pointing this out to them.

$$\text{(i)}\ f(\zeta) = \sum_{l=0}^{L} a_l \zeta^l, \tag{4.13}$$

$$\text{(ii)}\ f(\zeta) = b_0 + \sum_{l=1}^{L} b_l (\operatorname{Exp} \zeta^l - 1), \tag{4.14}$$

where a_l, b_l are constants. Expression (i) or equation (4.13) includes the model $f(\zeta) = \zeta^\delta$ suggested by Cowper and Symonds [8] as a special case. For $F(\zeta) = \zeta^\delta$, and $\delta \neq 1$, equation (4.12) becomes,

$$s - s_1 = \frac{\delta(1-\bar{c}^2)}{(1-\delta)\bar{c}} (g^{1-\delta} - g_1^{1-\delta}), \tag{4.15}$$

where $g_1 = g(s_1)$, s_1 is a constant, and β has been chosen as δ.

Another simple result is obtained for the constitutive relation (ii) or equation (4.14) with $L = 1$. The integrated expression is

$$s - s_1 = \frac{(1-\bar{c}^2)}{\bar{c}} \left[(g_1 - g) + \ln\left(\frac{e^g - 1}{e^{g_1} - 1}\right) \right], \tag{4.16}$$

where, again, $g_1 = g(s_1)$, s_1 is a constant, and β has been chosen as b_1.

It is of interest to note that for the constitutive relation (i), or equation (4.13), $f^{-1}(\zeta)$ may be expressed in the form:

$$f^{-1}(\zeta) = \sum_{k=1}^{m} c_k (\alpha_1 \zeta + \alpha_2)^{-k} + \sum_{p=1}^{n} (d_p \zeta + e_p)(\beta_1 \zeta^2 + 2\beta_2 \zeta + \beta_3)^{-p}, \tag{4.17}$$

where α_1, α_2, β_1, β_2, β_3, c_k, d_p, and e_p are real constants, with $\beta_2^2 < \beta_1 \beta_3$. Thus, the integral in equation (4.12) can always be evaluated in closed form in terms of elementary functions.

5. NONLINEAR WAVE MOTION BEHIND PROPAGATING DISCONTINUITIES

5.1 *Self-similar solution behind a constant-speed shockfront*

Consider a one-dimensional shockfront propagating at some speed $U(> 0)$ into an initially quiescent one-dimensional region ($x \geq 0$). As it had been remarked earlier, if the range of the rate of straining within the shock layer renders the material to exhibit instantaneous linear elasticity, then the shockfront will propagate at a constant speed,

$$U = \sqrt{E/\rho}, \tag{5.1}$$

where E is the instantaneous modulus of elasticity. If the value of E is chosen to be the same as the modulus of the elastic range of the constitutive equation, (2.6), then the shock speed has the same value as the characteristic speed $\mathrm{D}_+X/\mathrm{D}\bar{t}$ given by equation (4.1). Under such an assumption, the values of u and $\bar{\sigma}$ immediately behind the shock layer must satisfy the characteristic compatibility condition [10]:

$$\mathrm{d}\bar{\sigma} - \rho c\, \mathrm{d}u = -E\gamma f(\bar{\sigma}/\sigma_0 - 1)\, 1_+(\bar{\sigma}/\sigma_0 - 1)\, \mathrm{d}\bar{t}, \tag{5.2}$$

where $c = U = \sqrt{E/\rho}$.

The values of u, $\bar{\sigma}$, and ϵ, immediately behind the shock layer must also satisfy the jump conditions given by equations (2.13) to (2.15). Since u, $\bar{\sigma} = 0$ in the quiescent region in front of the shock layer, equation (2.14) requires that,

$$\bar{\sigma} = -\rho U u = -\rho c u, \tag{5.3}$$

immediately behind the shock layer. Combining equations (5.2) and (5.3), the following differential equation results:

$$2\beta \mathrm{d}\sigma = -f(\sigma) 1_+(\sigma)\, \mathrm{d}t, \tag{5.4}$$

where as before, $\sigma \equiv (\bar{\sigma}/\sigma_0 - 1)$ and $t \equiv \alpha c\bar{t}$ with $\alpha \equiv \beta\gamma\sqrt{\rho E/\sigma_0}$. Equation (5.4) indicates that, if $\sigma > 0$, then the overstress immediately behind the shockfront always attenuates with time along the shock.

Assuming that $\sigma > 0$ behind the shockfront, equation (5.4) can be integrated by quadrature as follows:

$$t = 2\beta \int^{\sigma} f^{-1}(\zeta)\, \mathrm{d}\zeta + C'', \tag{5.5}$$

where the integral is identical to that of equation (4.12). Thus, closed-form solutions of equation (5.5) are possible for special constitutive assumptions. For $f(\zeta) = \zeta^{\delta}$ with $\delta \neq 1$, equation (5.5) becomes

$$t = [2\delta/(1-\delta)]\,(\sigma_1^{1-\delta} - \sigma^{1-\delta}), \tag{5.6}$$

where $\sigma_1 \equiv \sigma(0)$, and β has been chosen as δ. Therefore, the overstress immediately behind the shock attenuates monotonically with time along the shockfront from $\sigma = \sigma_1$ at $t = 0$ to $\sigma = 0$ at $t = \infty$.

It is interesting to note that for $\delta = 2$, equation (5.6) can be satisfied by one of the self-similar solutions given in Section 3:

$$8\sigma^{-1} = (x+a)\,[2\eta + (\eta^2 - 1)\,\{A_1 + \ln\,[(\eta-1)/(\eta+1)]\}], \tag{5.7}$$

where $\eta = (t+a)/(x+a)$, and a, A_1 are constants. To satisfy the compatibility condition (5.6) for $\delta = 2$, the constant a in (5.7) must be chosen as follows:

$$a = 4/\sigma_1. \tag{5.8}$$

Therefore, the dimensionless overstress $\sigma(x, t)$ behind the shockfront is given by,

$$\sigma = 8/\left[(t + 4/\sigma_1)\left\{2 + (\eta - 1/\eta)\left[A_1 + \ln\left(\frac{\eta-1}{\eta+1}\right)\right]\right\}\right], \tag{5.9}$$

and the stress boundary condition at $x = 0$ is,

$$\sigma(0, t) = 8/[2(t + 4/\sigma_1) + (2t + \sigma_1 t^2/4)\,\{A_1 - \ln\,[1 + 8/(\sigma_1 t)]\}]. \tag{5.10}$$

The behavior of this function for various values of A_1 is shown in Fig. 2.

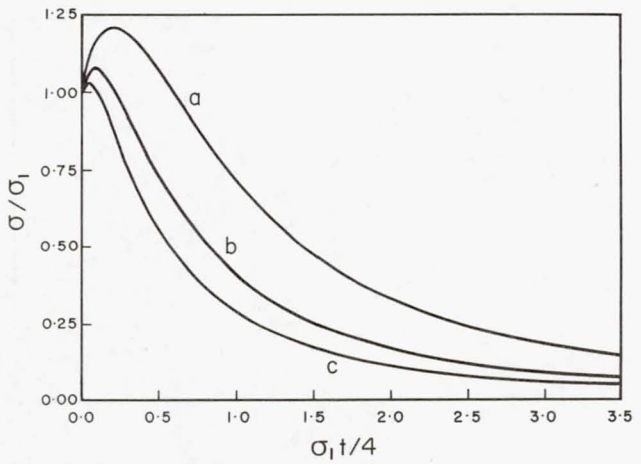

Fig. 2. Variation of σ/σ_1 with $\sigma_1 t/4$ at $\bar{X}=0$. Self-similar solution ($\delta=2$). (a) $A_1=\ln 2$, (b) $A_1=\ln 4$, (c) $A_1=\ln 8$.

In dimensional forms, the resulting expressions for $\sigma(\bar{X},\bar{t})$, $u(\bar{X},\bar{t})$, and $\epsilon(\bar{X},\bar{t})$ for this case are given as follows:

$$\sigma=\overline{\sigma}/\sigma_0-1=[4c\sigma_0/(E\gamma)](\bar{X}+ct_0)[2c(\bar{X}+ct_0)(\bar{t}+\bar{t}_0)-[c^2\bar{t}(\bar{t}+2t_0)-\bar{X}(\bar{X}+2ct_0)]\{\ln[(c\bar{t}+\bar{X}+2ct_0)/(c\bar{t}-\bar{X})]+B\}]^{-1}, \tag{5.11}$$

$$u=-(\sigma_0/\rho)[\sigma(\bar{t}+t_0)/(\bar{X}+ct_0)+c^{-1}], \tag{5.12}$$

$$\epsilon=(\sigma_0/\rho)[\sigma(\bar{t}+t_0)^2/(\bar{X}+ct_0)^2+c^{-2}], \tag{5.13}$$

where

$$t_0=2\sigma_0/(\gamma\sigma_1 E), \tag{5.14}$$

and B is a constant. The behavior of the functions $u(0,\bar{t})$ and $\epsilon(0,\bar{t})$ for various values of A_1 are indicated in Figs. 3 and 4. It is of interest to note that (5.13) yields a permanent strain ϵ_p given by

$$\epsilon_p=\lim_{\bar{t}\to\infty}\epsilon=(\sigma_0/E)[(4c\sigma_0/A_1\gamma E))(\bar{X}+ct_0)^{-1}+1]. \tag{5.15}$$

Typical distributions of the permanent strain are shown in Fig. 5.

5.2. *Constant speed solution behind an elastic precurser*

Duvall [23] suggested that the one-dimensional, unsteady motion in a semi-infinite ($x\geqslant 0$), rate-sensitive, elastoplastic region generated by a continuously applied load at its boundary ($x=0$) may eventually consist of an elastic precurser propagating into an initially quiescent region, an unloading, relaxation zone, and a sub-elastic, constant-speed, nonlinear wave motion as depicted in Fig. 6. After a reasonable length of time, the elastic precurser will be far ahead of the leading wave of the constant speed region

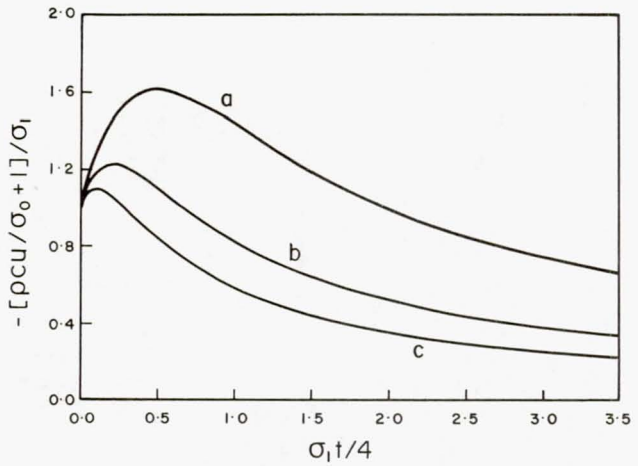

Fig. 3. Variation of $-[\rho cu/\sigma_0+1]/\sigma_1$ with $\sigma_1 t/4$ at $\bar{X}=0$. Self-similar solution ($\delta=2$). (a) $A_1=\ln 2$, (b) $A_1=\ln 4$, (c) $A_1=\ln 8$.

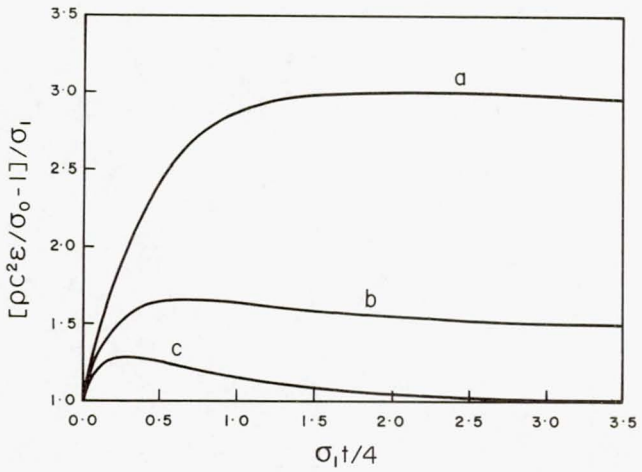

Fig. 4. Variation of $[\rho c^2\epsilon/\sigma_0-1]/\sigma_1$ with $\sigma_1 t/4$ at $\bar{X}=0$. Self-similar solution ($\delta=2$). (a) $A_i=\ln 2$, (b) $A_1=\ln 4$, (c) $A_1=\ln 8$.

and the boundary of $x=0$ will be far behind it. Relative to an observer moving with the leading wave of the constant speed region, the unsteady motion behind the leading wave becomes essentially steady.

Any of the constant-speed, nonlinear wave solutions described by equation (4.12) in section 4 with $\bar{c}<1$ may be considered as a constant-speed portion of such an 'unsteady' motion. If the overstress on the leading wave $s=s_1$ is $\sigma_1>0$, then the quadrature expression, (4.12), becomes,

$$s-s_1=[\beta(1-\bar{c}^2)/\bar{c}]\int_{\sigma_1}^{\sigma} f^{-1}(\zeta)\,\mathrm{d}\zeta, \tag{5.16}$$

where $s\equiv\bar{c}t-x$, and $\bar{c}<1$. From equation (5.16), the stress boundary condition

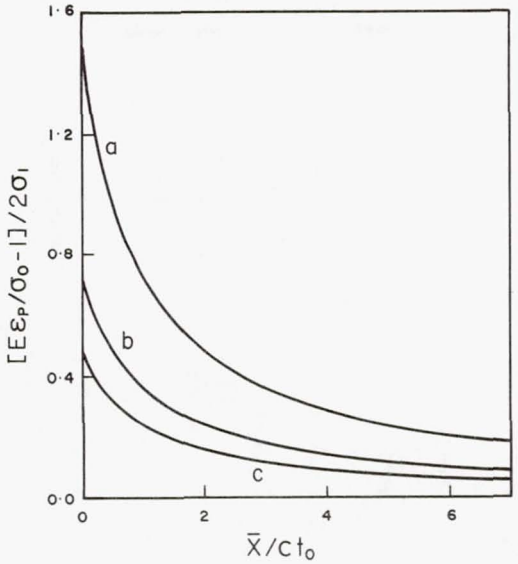

Fig. 5. Typical distributions of the permanent strain. Self-similar solution ($\delta = 2$). (a) $A_1 = \ln 2$, (b) $A_1 = \ln 4$, (c) $A_1 = \ln 8$.

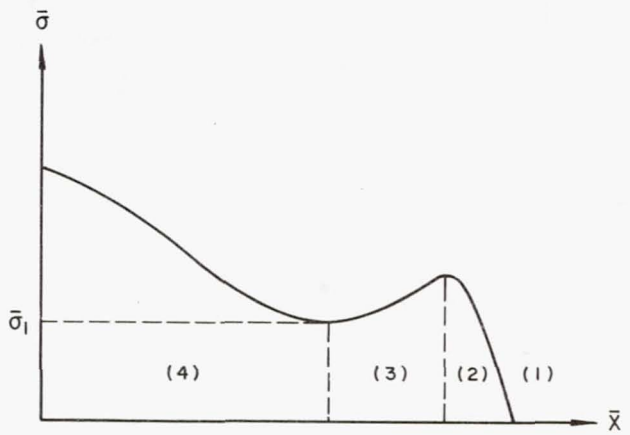

Fig. 6. Schematic representation of a nonlinear wave motion behind a relaxation zone. (1) quiescent region, (2) elastic precursor, (3) relaxation zone, (4) sub-elastic, constant speed region.

required to maintain the constant-speed motion can be evaluated in a straightforward manner.

It is of interest to note that, if the initial rate of loading of the applied stress at the boundary is not too high such that the time required to raise the stress $\bar{\sigma}$ from zero to the value of the static yield stress σ_0 is much longer than the pertinent characteristic relaxation time of the medium, then a complete description of a possible non-linear wave motion for a continuously loading boundary may be constructed exactly. Figure 7 is a schematic representation of such a motion. It consists of four solution regions

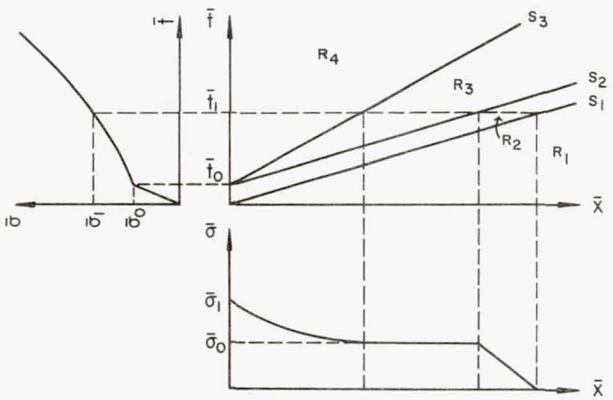

Fig. 7. Schematic representation of a possible nonlinear wave motion due to a continuously loading boundary at $\bar{X} = 0$.

separated by three discontinuities described as follows:

(1) *Solution regions*

R_1: the undisturbed region, $\bar{\sigma} = 0$

R_2: the elastic region, $\bar{\sigma} < \sigma_0$

R_3: the constant stress region, $\bar{\sigma} = \sigma_0$

R_4: the constant speed solution region, $\bar{\sigma} > \sigma_0$.

(2) *Discontinuities*

S_1: the leading elastic wave

S_2: the trailing elastic wave

S_3: the leading constant speed wave.

Such an unsteady motion may be generated by a monotonically increasing stress boundary condition. The manner in which the stress varies at the boundary in the elastic range can be quite arbitrary (so long as the rate of loading is small enough so that there will be no dynamic overstressing in the elastic precurser) and has been chosen as a linear function of $\bar{t}$ in Fig. 7 for simplicity, while the rate of stressing beyond the static yield stress must follow the expression given in equation (4.12) or the differential equation, (4.7). It is clear from equation (4.7) that nontrivial solutions in R_4 can be generated from a leading wave S_3 on which the stress is $\bar{\sigma} = 0$ only if $f(0) > 0$. The value of $f(0)$ can be arbitrarily small. Alternatively, if $f(0) = 0$ for a specific constitutive relation, a solution such as the one depicted by Fig. 7 can still be generated by viewing S_3 as a small discontinuity in the value of $\bar{\sigma}$ such that the overstress jumps across this 'plastic shock' from zero to some small constant positive value $\sigma_1 \ll 1$. The jump in σ from 0 to σ_1 at the boundary of $\bar{X} = 0$ may be viewed as the result of a very fast rate of loading r in a small interval of time $\Delta\bar{t}$ near $\bar{t} = \bar{t}_0$ such that $\lim_{\Delta\bar{t} \to 0} (r\Delta\bar{t}) = \bar{\sigma}_1 = (1 + \sigma_1)\sigma_0$ and $0 < \sigma_1 \leqslant 1$. The fact that $\bar{\sigma} = \sigma_0$ in R_3 is an admissible solution can be deduced immediately from equation (2.7).

Acknowledgment—This investigation was partially supported by the Advanced Research Projects Agency of the Department of Defense and monitored by the Office of Naval Research under Contract Number N00014-68-A-0187, and partially supported by the National Aeronautics and Space Administration under Grant Number NASA-NGL34-002-084. A portion of this work was completed when one of us (T. S. C.) was visiting the Center for the Application of Mathematics, Lehigh University, and the Department of

Theoretical and Applied Mechanics, Cornell University. The authors are indebted to Professors A. C. Eringen, Y. Horie, G. S. S. Ludford, R. S. Rivlin, and E. Varley for their comments and suggestions.

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(*Received* 20 *July* 1971)

Résumé—Deux classes de solutions du type fermé d'ondes non-linéaires à une dimension, dans un matériau élasto-plastique sensible àla vitesse sont présentées. Une classe de ces solutions est auto-semblable et l'autre classe consiste en des propagations à vitesse constante. Des applications de ces solutions à des mouvements instables en arrière de la propagation de discontinuités sont également considérées.

Zusammenfassung—Zwei Klassen von Lösungen geschlossener Form von eindimensionalen, nicht-linearen Wellen in einem ratenempfindlichen elestoplastischen Stoffe werden mitgeteilt. Eine Klasse dieser Lösungen ist selbst-ähnlich und die andere Klasse besteht aus konstanten Geschwindigkeitsfortpflanzungen. Anwendungen dieser Lösungen auf unstetige Bewegungen hinter sich fortpflanzenden Diskontinuitäten werden auch untersucht.

Sommario—Si tratta di una relazione su due classi di soluzioni di forma chiusa di onde monodimensionali e non lineari in un materiale elastoplastico e sensibile al ritmo. Una classe di queste soluzioni è autosimilare e l'altra consiste in propagazioni a velocità constante. Si prendono anche in considerazione le applicazioni di queste soluzioni ai movimenti instabili dietro le discontinuità di propagazione.

Абстракт — Установлены два класса решений замкнутого вида для одномерных нелинейных волн в эластопластическим материале чувствительном к скорости. Один класс этих решений — самоподобным, а другой состоит из распространений постоянной скорости. Рассмотрены и применения эти решений к нестационарным движениям за распространяющиесь разрывы.

THE PHYSICS OF FLUIDS VOLUME 15, NUMBER 3 MARCH 1972

Research Notes

Research Notes published in this Section include important research results of a preliminary nature which are of special interest to the physics of fluids and new research contributions modifying results already published in the scientific literature. Research Notes cannot exceed five printed columns in length including space allowed for title, abstract, figures, tables, and references. The abstract should have three printed lines. Authors must shorten galley proofs of Research Notes longer than five printed columns before publication.

Curved Characteristics Behind Blast Waves

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(Received 22 December 1970; final manuscript received 6 August 1971)

Exact solutions, expressed in closed form in terms of elementary functions, are presented for the three sets of curved characteristics behind a self-similar, strong blast wave.

Blast waves are produced in gaseous media due to the sudden deposition of large amounts of energy in relatively small regions. The propagation of a point-source blast wave into an ideal gas, whose initial pressure is assumed to be negligibly low, is known to be self-similar. This property can be deduced using either dimensional arguments[1] or invariant theorems of continuous groups of transformations.[2] Closed form solutions describing the flow variables in the nonisentropic region behind such a blast wave in $n=1$, 2, 3 dimensions were obtained independently by von Neumann[3] and Sedov.[4] The reflection of strong blast waves was discussed in an earlier paper[5] by the authors.

It is known that the basic equations governing the nonisentropic flow behind a propagating blast wave admit three distinct characteristic directions given by

$$\frac{dr}{dt}=u, \qquad u\pm a, \tag{1}$$

where r is the radial distance from the point of explosion, t is the elapsed time, u is the velocity, and a is the isentropic speed of sound. The first characteristic direction coincides with the local particle velocity and the family of characteristics is, therefore, composed of particle lines. This direction corresponds to the speed of propagation of entropy disturbances. The other two characteristic directions correspond to the local speeds of propagation of pressure, density, or velocity disturbances. The purpose of this note is to report the interesting result that these three families of curved characteristics can also be represented in closed form, and remarkably, still in terms of elementary functions.

First, consider the (u) characteristics or particle lines. According to its definition and the self-similar solution of Ref. 5, it may be easily deduced that along a (u) characteristic,

$$dr/dt=(2+n)\,U\hat{u}y(\hat{u})/2, \tag{2}$$

where

$$U=[2K_n/(2+n)](E_0/\rho_0)^{1/(2+n)}t^{-n/(2+n)} \tag{3}$$

is the shock speed, $y(\hat{u})$ is given by Eq. (5) of Ref. 5, $\hat{u}=ut/r$, E_0 is the energy released per unit area for a planar wave, per unit length for a cylindrical wave, and the total energy released for a spherical wave, ρ_0 is the initial density, and K_n is determined by the condition of conservation of total energy.

But, from the definition of the similarity parameter and the expression for the shock radius of Ref. 5, it can be shown that, in general,

$$dr/dt=K_n(E_0/\rho_0)^{1/(2+n)}t^{-n/(2+n)}[2y/(2+n)+t\,dy/dt]. \tag{4}$$

Therefore, from Eq. (5) of Ref. 5 and Eqs. (2)–(4), it is found that,

$$d\ln t=\frac{d\ln y}{d\hat{u}}\frac{d\hat{u}}{\hat{u}-[2/(2+n)]}, \tag{5}$$

along a particle line, where,

$$\frac{d\ln y}{d\hat{u}}=\frac{(\gamma-1)\hat{u}^2/2+[\hat{u}-2/(2+n)][\hat{u}-2/\gamma(2+n)]}{\hat{u}[2/\gamma(2+n)-\hat{u}][(2-n+n\gamma)\hat{u}/2-1]}, \tag{6}$$

and γ is the adiabatic index. Thus, Eq. (5) can be integrated in terms of elementary functions. The result is

$$\hat{t}=\alpha\hat{u}(A_2-\hat{u})^{\beta_1}(\hat{u}-A_2/\gamma)^{\beta_2}(\hat{u}-A_5^{-1})^{\beta_3}, \tag{7}$$

where

$$\alpha=A_1(A_2-A_1^{-1})^{-\beta_1}(A_1^{-1}-A_2/\gamma)^{-\beta_2}(A_1^{-1}-A_5^{-1})^{-\beta_3},$$

$$\beta_1=(2\gamma+n-2)[n(\gamma-2)]^{-1}\beta_2,$$

$$\beta_2=n\gamma(2+n)(2-\gamma)(\gamma-1)[2(2n\gamma+2\gamma-n+2)\times(2-n-2\gamma)+2n\gamma(2-\gamma)(n\gamma+4)+2(\gamma+1)(2+n)\times(n\gamma^2-2n\gamma+n+2\gamma-2)]^{-1},$$

$$\beta_3=(2+n)(n\gamma^2-2n\gamma+n+2\gamma-2)[n(2-\gamma)\times(n\gamma-n+2)]^{-1}\beta_2-1, \quad (8)$$

$\hat{t}=t/t_0$ is a dimensionless time normalized with respect to some convenient time scale t_0, and the A's are given by Eqs. (6) of Ref. 5.

To complete the solution for the particle lines, an expression relating $(r, t, \hat{u})$ is obtained from Eqs. (2) and (5) of Ref. 5. In dimensionless form, this expression becomes,

$$\hat{r}=(A_1\hat{u})^{-A_2}[A_3(\gamma\hat{u}/A_2-1)]^{A_8}\times[A_1(1-A_5\hat{u})/(A_1-A_5)]^{A_6-A_9\hat{t}^{2/(2+n)}}, \quad (9)$$

where $\hat{r}=r/R(t_0)$. Equations (8) and (9) form the closed form parametric solution for the family of (u) characteristics or particle lines behind a self-similar blast wave.

For the $(u\pm a)$ characteristics, we have

$$\frac{d_\pm r}{dt}=u'f\pm\left[\frac{\gamma p'g}{\rho' h}\right]^{1/2}=[U/(\gamma+1)]\{2A_1\hat{u}y(\hat{u})\pm[2\gamma(\gamma-1)g(\hat{u})/h(\hat{u})]^{1/2}\}, \quad (10)$$

where $g(\hat{u})$, $h(\hat{u})$, and $y(\hat{u})$ are given by Eqs. (4) and (5) of Ref. 5, and the shock speed U by Eq. (3). Combining with Eq. (4) and after considerable manipulation, it is found that

$$d\ln t=G(\hat{u})\,d\hat{u}, \quad (11)$$

where,

$$G(\hat{u})=(d\ln y/d\hat{u})\{[\hat{u}-2/(2+n)]\pm[\gamma(\gamma-1)(2-2\hat{u}-n\hat{u})/2(2\gamma\hat{u}+n\gamma\hat{u}-2)]^{1/2}\hat{u}\}^{-1}. \quad (12)$$

The expression $d\ln y/d\hat{u}$ in $G(\hat{u})$ is known and given by Eq. (6). Therefore, $G(\hat{u})$ may be reduced to an algebraic function involving square roots of second degree polynomials of $\hat{u}$ as radicals. This means that Eq. (11) can be integrated in terms of elementary, albeit transcendental, functions. The results are

$$\ln\hat{t}=K_\pm+\ln|A_5\hat{u}/(1-A_5\hat{u})|\pm(m/l^{1/2})\times\sin^{-1}\{[l/(A_5^{-1}-\hat{u})-k]/(k^2-l)^{1/2}\}, \quad (13)$$

where,

$$k=A_5^{-1}-(\gamma+1)[\gamma(2+n)]^{-1},$$

$$l=A_5^{-2}-2A_5^{-1}(\gamma+1)[\gamma(2+n)]^{-1}+4[\gamma(2+n)^2]^{-1},$$

$$m=-A_5^{-1}[(\gamma-1)/2]^{1/2}. \quad (14)$$

Equation (13) and the parametric expression for $(\hat{r}, \hat{t}, \hat{u})$ given by Eq. (9) form the closed form parametric solutions for the two families of $(u\pm a)$ characteristics in the dimensionless $\hat{r}$–$\hat{t}$ plane. The constant $K_\pm$ in Eq. (13) must be evaluated for each characteristic from a set of known values of $(\hat{r}, \hat{t}, \hat{u})$.

Figure 1 displays a typical set of solution curves expressing the dimensionless time $\hat{t}$ as functions of the dimensionless distance $\hat{r}$ for $n=3$ and $\gamma=\frac{5}{3}$. The terminating curve A is the path of the front of the blast wave. Behind A, there are three families of characteristics: (1) The solid curves are the (u) characteristics or particle lines; (2) the dashed curves are the $(u+a)$ characteristics; and (3) the dot-dashed curves are the $(u-a)$ characteristics.

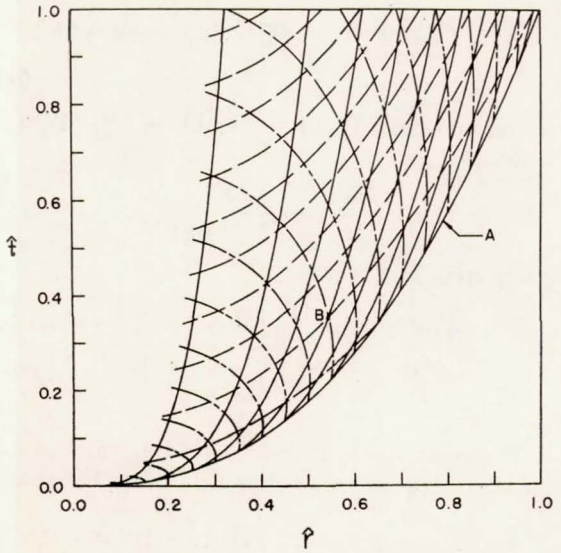

FIG. 1. Curved characteristics behind a strong blast wave for $n=3$, $\gamma=5/3$.

The authors are grateful to Mr. T. Tucker of Oak Ridge National Laboratory for computational assistance.

This research was supported in part by the Department of Defense Project Themis at Lehigh University under Contract Number DAAD05-69-C-0053 monitored by the Ballistic Research Laboratories, and by the National Aeronautics and Space Administration Grant number NASA-NGL34-002-084.

A portion of this manuscript was completed when one of us (T.S.C.) was visiting Cornell University under the auspices of the Army Research Office, Durham.

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THE PHYSICS OF FLUIDS VOLUME 16, NUMBER 1 JANUARY 1973

Comments on "Application of singular eigenfunction expansions to the propagation of periodic disturbances in a radiating grey gas"

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(Received 5 June 1972)

Recently, we have been extending our analysis[1] of sound propagation in dissociative, radiative media to include the effects of scattering. It is of interest to note that the set of pertinent integrodifferential equations can also be solved exactly using the singular eigenfunction expansion technique. For the special case of an ideal, grey, scattering gas, our resulting integral equation for the elementary solutions (in terms of the notations of Cheng and Leonard[2]) becomes

$$\left(1-\frac{\mu}{\mathrm{Bu}\nu}\right)\Psi_\nu(\mu)=\tfrac{1}{2}\{[\bar{g}(\nu;c)]^{-1}+c\}\int_{-1}^{1}\Psi_\nu(\mu')\,d\mu', \tag{1}$$

with

$$\bar{g}(\nu;c)=\frac{[(1-c)+i\beta](\nu^2-\bar{\xi}^2)}{(\nu^2+1/\gamma)(1-c)^2},$$

and

$$\bar{\xi}^2=-\frac{(1-c)/\gamma+i\beta}{(1-c)+i\beta},$$

where c denotes the single scattering albedo which is the ratio of the scattering coefficient to the extinction coefficient.

It can be shown that Eq. (1) admits both discrete eigenvalues and a continuous spectrum. The discrete eigenvalues are determined by the dispersion relation

$$\Lambda(\nu)=1-\tfrac{1}{2}(\mathrm{Bu}\nu)\{[\bar{g}(\nu;c)]^{-1}+c\}\int_{-1}^{1}\frac{d\mu}{\mathrm{Bu}\nu-\mu}=0, \tag{2}$$

which for $c=0$ corresponds to Eq. (11) of Ref. 2. Using the argument principle we are able to show that Eq. (2) admits either two or four discrete roots in plus-minus complex pairs. In Figs. 1 and 2 we present the discrete eigenvalues of ν_0 and ν_1 for Bo (Boltzmann number) = 3.23, $\gamma=1.4$, and $c=0.2$. We note that, similar to the results of Cheng and Leonard, there exists a frequency cutoff of the second pair of roots $\pm\nu_1$ below a critical Bouguer number, $\mathrm{Bu}\cong 0.08161$.

We find that it is possible to obtain explicit expressions for the eigenroots of the transcendental equation (2). For example, to determine the explicit expressions for the four roots for $c=0$ with $\mathrm{Bu}\geq\mathrm{Bu}_1=0.1107$, we merely calculate $\Lambda(z)$ and $X(\pm z)$ at two arbitrary values of $z=z_{1,2}$ (preferably along the imaginary axis) according to Eqs. (12) and (26) of Ref. 2. Equation (34) of Ref. 2 then provides a pair of coupled polynomial equations for ν_0 and ν_1,

$$(\nu_0^2-z_i^2)(\nu_1^2-z_i^2)=\frac{\Lambda(z_i)(\xi^2-z_i^2)[1+(1/i\beta)]}{X(z_i)X(-z_i)}$$

for $i=1$ and 2. (3)

Equations (3) can be solved readily to yield the desired explicit expressions for $\pm\nu_0$ and $\pm\nu_1$. Results of some

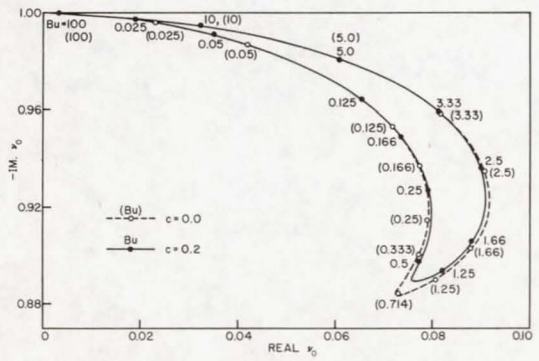

FIG. 1. The first discrete root (ν_0) of $\Lambda(\nu)=0$ for Bo = 3.23, $\gamma=1.4$, and $c=0.0$, 0.2.

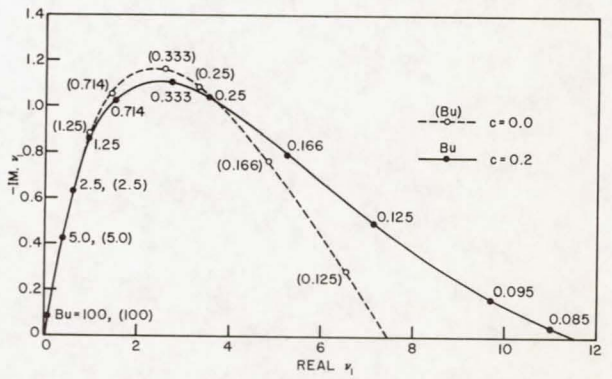

FIG. 2. Second discrete root (ν_1) of $\Lambda(\nu)=0$ for Bo = 3.23, $\gamma=1.4$, and $c=0.0$, 0.2.

TABLE I. Discrete eigenvalues for Bu=1/6, Bo=3.23, $\gamma=1.4$, and $c=0$.

	a	b	c
$\mathrm{Re}\nu_0$	0.07738627380	0.07738629373	0.07738
$\mathrm{Im}\nu_0$	−0.9372360013	−0.9372359958	−0.9372
$\mathrm{Re}\nu_1$	4.860766870	4.860766867	4.860
$\mathrm{Im}\nu_1$	−0.7599477400	−0.7599477363	−0.7599

[a] Calculated using a bisection method where the roots are determined by simultaneous sign changes of the real and imaginary parts of $\Lambda(z)$.

[b] Calculated using the explicit expressions.

[c] Values provided by P. Cheng.

sample calculations of the discrete eigenvalues based on this method have been compared with those obtained using the more conventional numerical schemes in Table I. In general, we are able to obtain an accuracy up to eight significant figures using the explicit expressions.

The authors are indebted to Professor P. Cheng for his assistance and for providing us with the numerical values quoted in Table I.

We are grateful to Dr. E. E. Burniston and Dr. C. E. Siewert for valuable discussion and to Dr. J. T. Kriese for his help in calculating the eigenroots using the explicit scheme.

This research was supported in part by National Aeronautics and Space Administration grant NGL34-002-084 and National Science Foundation grant GU-1590.

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[redacted]

PERGAMON PRESS
OXFORD · NEW YORK

Int. J. Engng Sci., 1974, Vol. 12, pp. 441–470. Pergamon Press. Printed in Great Britain

ELEMENTARY SOLUTIONS OF COUPLED MODEL EQUATIONS IN THE KINETIC THEORY OF GASES

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Abstract—The method of elementary solutions is employed to solve two coupled integrodifferential equations sufficient for determining temperature–density effects in a linearized BGK model in the kinetic theory of gases.

Full-range completeness and orthogonality theorems are proved for the developed normal modes and the infinite-medium Green's function is constructed as an illustration of the full-range formalism.

The appropriate homogeneous matrix Riemann problem is discussed, and half-range completeness and orthogonality theorems are proved for a certain subset of the normal modes. The required existence and uniqueness theorems relevant to the **H** matrix, basic to the half-range analysis, are proved, and an accurate and efficient computational method is discussed. The half-space temperature-slip problem is solved analytically, and a highly accurate value of the temperature-slip coefficient is reported.

1. INTRODUCTION

THERE EXISTS in the kinetic theory of gases a class of one-dimensional problems for which the transverse momentum and heat-transfer effects can be separated by projecting the basic kinetic equation describing the particle distribution function onto certain properly chosen directions in a Hilbert space. The resulting expression describing the heat-transfer and compressibility effects is a vector integrodifferential equation with a matrix kernel similar in form to one studied previously by Bond and Siewert [4] and Burniston and Siewert [5] in connection with the scattering of polarized light. It can be shown that such a vector integrodifferential equation admits a general solution similar to that suggested by Case [7] for scalar transport problems and applied by Cercignani [12] to kinetic equations.

We develop in this paper the elementary solutions to the vector integrodifferential equation basic to a linearized, constant collision frequency (BGK) model suggested by Bhatnagar *et al.* [3] and Welander [29]. The elementary solutions, some of which are generalized functions [14], can be shown to possess rather general full-range and half-range completeness and orthogonality properties. The expansion (or completeness) theorems are proved by reducing a system of singular integral equations to an equivalent matrix Riemann problem and subsequently making use of the theory of Mandžavidze and Hvedelidze [20] and Muskhelishvili [21] to establish the solubility of the resulting equations.

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As an application of our established analysis, we construct in this paper the infinite-medium Green's function useful for developing particular solutions to the basic transport equation. We also make use of the half-range expansion theorem to solve the notoriously difficult temperature slip problem considered previously [2, 19, 23, 28, 29] by approximate methods. Our solution permits an accurate computation of the 'temperature slip coefficient' which may be used to evaluate the merits of approximate techniques.

2. THE KINETIC MODEL AND LINEARIZATION

Basically, the BGK model is constructed by replacing the collision integral in the Boltzmann equation by a more tractable relaxation term; we therefore write

$$\left[\frac{\partial}{\partial\tau}+\mathbf{u}\cdot\nabla\right]f(\mathbf{y},\mathbf{u},\tau)=\nu[\hat{f}(\mathbf{y},\mathbf{u},\tau)-f(\mathbf{y},\mathbf{u},\tau)], \tag{2.1}$$

where $f(\mathbf{y},\mathbf{u},\tau)$ is the particle distribution function, $\mathbf{y}$ is the position vector, $\mathbf{u}$ is the particle velocity, τ is the time, and ν is a characteristic collision frequency. To ensure that the model conserves particles, momentum and energy, we require that

$$\int[\hat{f}(\mathbf{y},\mathbf{u},\tau)-f(\mathbf{y},\mathbf{u},\tau)]\mathbf{U}\,\mathrm{d}^3u=\mathbf{0}, \tag{2.2}$$

where the integration is to be taken over all velocity space and $\mathbf{U}$ is a five-element vector with components 1, u_1, u_2, u_3, and u^2, the collisional invariants. Here u_α, $\alpha=1, 2,$ and 3, and u are respectively the components and magnitude of $\mathbf{u}$. The invariance requirements given by equation (2.2) can be satisfied by choosing

$$\hat{f}(\mathbf{y},\mathbf{u},\tau)=n(\mathbf{y},\tau)\left[\frac{m}{2\pi kT(\mathbf{y},\tau)}\right]^{3/2}\exp\left[-\frac{m|\mathbf{u}-\mathbf{q}(\mathbf{y},\tau)|^2}{2kT(\mathbf{y},\tau)}\right], \tag{2.3}$$

the local Maxwellian distribution. Here m is the particle mass and k is the Boltzmann constant. In addition

$$\begin{bmatrix} n(\mathbf{y},\tau) \\ n(\mathbf{y},\tau)\,\mathbf{q}(\mathbf{y},\tau) \\ 3n(\mathbf{y},\tau)\,kT(\mathbf{y},\tau)\end{bmatrix}=\int f(\mathbf{y},\mathbf{u},\tau)\begin{bmatrix} 1 \\ \mathbf{u} \\ m|\mathbf{u}-\mathbf{q}(\mathbf{y},\tau)|^2\end{bmatrix}\mathrm{d}^3u \tag{2.4}$$

defines the local number density $n(\mathbf{y},\tau)$, the fluid velocity $\mathbf{q}(\mathbf{y},\tau)$, and the absolute temperature $T(\mathbf{y},\tau)$.

It is not difficult to demonstrate that the model given by equations (2.1), (2.3), and (2.4) admits an H theorem, such that

$$\frac{\partial}{\partial\tau}\int f(\mathbf{u},\tau)\ln f(\mathbf{u},\tau)\,\mathrm{d}^3u\leqslant 0, \tag{2.5}$$

for spatially uniform conditions. Thus the model possesses many of the important properties of the full Boltzmann equation.

Because of equation (2.4), the model is described by a nonlinear functional equation; however, we consider circumstances for which the particle distribution function

$f(\mathbf{y}, \mathbf{u}, \tau)$ differs only slightly from an initial Maxwellian distribution $f_0(\mathbf{u})$ characterized by a set of constant initial values of the number density n_0, fluid velocity $\mathbf{q}_0$, and temperature T_0. If we now write

$$f(\mathbf{y}, \mathbf{u}, \tau) = f_0(\mathbf{u}) + f_1(\mathbf{y}, \mathbf{u}, \tau), \tag{2.6}$$

and truncate $\hat{f}(\mathbf{y}, \mathbf{u}, \tau)$ at the linear terms in a Taylor series expansion about f_0, we find that equation (2.1) can be approximated by

$$\left[\frac{\partial}{\partial t} + (\mathbf{c} + \mathbf{v}) \cdot \nabla + 1\right] \hat{h}(\mathbf{x}, \mathbf{c}, t) = \int \hat{h}(\mathbf{x}, \mathbf{c}', t) K(\mathbf{c}' : \mathbf{c}) \mathrm{e}^{-c'^2} \mathrm{d}^3 c', \tag{2.7}$$

with

$$K(\mathbf{c}' : \mathbf{c}) = \frac{1}{\pi^{3/2}}\left[1 + 2\mathbf{c} \cdot \mathbf{c}' + \frac{2}{3}\left(c^2 - \frac{3}{2}\right)\left(c'^2 - \frac{3}{2}\right)\right], \tag{2.8}$$

and where

$$\mathbf{x} = \nu\left(\frac{m}{2kT_0}\right)^{1/2} \mathbf{y}, \qquad t = \nu\tau, \tag{2.9a,b}$$

$$\mathbf{c} = \left(\frac{m}{2kT_0}\right)^{1/2} (\mathbf{u} - \mathbf{q}_0), \qquad \mathbf{v} = \left(\frac{m}{2kT_0}\right)^{1/2} \mathbf{q}_0, \tag{2.10a,b}$$

and

$$f_0(\mathbf{c}) \hat{h}(\mathbf{x}, \mathbf{c}, t) = f_1(\mathbf{x}, \mathbf{c}, t). \tag{2.11}$$

A model equation more general than equation (2.7) may be constructed, as suggested by Gross and Jackson [13] and Sirovich [27], by expanding the kernel of a linearized Boltzmann equation in an appropriately chosen complete and orthonormal set of eigenfunctions. We shall, however, restrict our attention to the linearized model described by equation (2.7).

If we now let $\phi_\alpha(\mathbf{c})$, $\alpha = 1, 2, \ldots, 5$, denote the elements of the vector

$$\boldsymbol{\phi}(\boldsymbol{c}) = \frac{1}{\pi^{3/4}} \begin{bmatrix} 1 \\ \sqrt{\frac{2}{3}}(c^2 - \frac{3}{2}) \\ \sqrt{2}\, c_2 \\ \sqrt{2}\, c_3 \\ \sqrt{2}\, c_1 \end{bmatrix}, \tag{2.12}$$

where c_1, c_2, c_3, and c are respectively the components and magnitude of $\mathbf{c}$, then equation (2.8) can be written as

$$K(\mathbf{c}' : \mathbf{c}) = \tilde{\boldsymbol{\phi}}(\mathbf{c}) \boldsymbol{\phi}(\mathbf{c}'). \tag{2.13}$$

Here the superscript tilde is used to denote the transpose operation. We note that the

elements of $\boldsymbol{\phi}(\mathbf{c})$ obey the orthonormal conditions

$$(\phi_\alpha, \phi_\beta)_a = \delta_{\alpha,\beta}; \quad \alpha,\beta = 1, 2, \ldots, 5, \tag{2.14}$$

in a Hilbert space (a) of the functions of $\mathbf{c}$ defined by the inner product

$$(A_1, A_2)_a = \int A_1(\mathbf{c}) A_2(\mathbf{c})\, \mathrm{e}^{-c^2}\, \mathrm{d}^3 c. \tag{2.15}$$

The elements $\phi_\alpha(\mathbf{c})$ are, of course, related to the collisional invariants which define the $\mathbf{U}$ vector in equation (2.2), and the orthogonality conditions stated in equation (2.14) are therefore direct consequences of the invariance requirements of equation (2.2).

3. THE VECTOR KINETIC EQUATION

As stated in the Introduction, we are primarily interested in steady-state gas-kinetic problems with plane symmetry. Without loss of generality, we set $\mathbf{q}_0 = \mathbf{0}$, and thus the steady-state version of equation (2.7) for

$$h(x_1, \mathbf{c}) \triangleq \hat{h}(x_1, \mathbf{c}) - 2\pi^{-3/2} c_1 (c_1, \hat{h})_a \tag{3.1}$$

becomes

$$\left[c_1 \frac{\partial}{\partial x_1} + 1 \right] h(x_1, \mathbf{c}) = \sum_{\alpha=1}^{4} \phi_\alpha(\mathbf{c})(\phi_\alpha, h)_a. \tag{3.2}$$

We now follow Cercignani[12] and consider the functions

$$\begin{aligned} &g_1(c_2, c_3) = \pi^{-1/2}, \qquad g_2(c_2, c_3) = \pi^{-1/2}(c_2^2 + c_3^2 - 1), \\ &g_3(c_2, c_3) = \left(\frac{\pi}{2}\right)^{-1/2} c_2, \quad \text{and} \quad g_4(c_2, c_3) = \left(\frac{\pi}{2}\right)^{-1/2} c_3. \end{aligned} \tag{3.3}$$

It is a straightforward matter to demonstrate that the g functions given by equations (3.3) satisfy the orthonormal conditions

$$(g_\alpha, g_\beta)_b = \delta_{\alpha,\beta}, \; \alpha = 1, 2, 3, \text{and } 4, \tag{3.4}$$

in a subspace (b) of the functions of c_2 and c_3 defined by the inner product

$$(B_1, B_2)_b = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} B_1(c_2, c_3) B_2(c_2, c_3) \mathrm{e}^{-(c_2^2 + c_3^2)}\, \mathrm{d}c_2 \mathrm{d}c_3. \tag{3.5}$$

We now span the Hilbert space (b) by the subspace (c) characterized by the g_α's and a subspace (d), the orthogonal complement to (c); subsequently we expand $h(x_1, \mathbf{c})$ of equation (3.2) in the manner

$$h(x_1, \mathbf{c}) = \sum_{\alpha=1}^{4} \Psi_\alpha(x_1, c_1) g_\alpha(c_2, c_3) + \Psi_5(x_1, \mathbf{c}), \tag{3.6}$$

where $\Psi_5(x_1, \mathbf{c})$ is the component of $h(x_1, \mathbf{c})$ belonging to the subspace (d). Such an

expansion yields the interesting property that the inner products $(\Psi_\alpha, \phi_\alpha)_a$, $\alpha = 1, 2, 3$, and 4, are simply related to the perturbations of the number density, the temperature, and the transverse components of the fluid velocity, respectively.

Substituting equation (3.6) into equation (3.2) and projecting each term onto the appropriate directions of g_α in the Hilbert space (b), we obtain

$$\left[\mu\frac{\partial}{\partial x}+1\right]\hat{\boldsymbol{\Psi}}(x,\mu)=\frac{1}{\sqrt{\pi}}\mathbf{J}(\mu)\int_{-\infty}^{\infty}\tilde{\mathbf{J}}(\mu')\hat{\boldsymbol{\Psi}}(x,\mu')\,e^{-\mu'^2}\,d\mu', \tag{3.7}$$

where $\hat{\boldsymbol{\Psi}}(x, \mu)$ is a four-element vector with components $\Psi_\alpha(x, \mu)$, $\alpha = 1, 2, \ldots, 4$, and

$$\mathbf{J}(\mu)=\begin{bmatrix}\sqrt{\frac{2}{3}}(\mu^2-\frac{1}{2}) & 1 & 0 & 0\\ \sqrt{\frac{2}{3}} & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}. \tag{3.8}$$

For convenience, we have changed the variables x_1 and c_1 to x and μ. We note that the two functions $\Psi_1(x, \mu)$ and $\Psi_2(x, \mu)$, characterizing the perturbations of the number density and temperature respectively, are described by a set of two coupled integrodifferential equations. These two equations are, of course, uncoupled from the functions $\Psi_3(x, \mu)$ and $\Psi_4(x, \mu)$ which describe the perturbations of the transverse momenta.

4. ELEMENTARY SOLUTIONS OF THE TWO-VECTOR TRANSPORT EQUATION RELEVANT TO TEMPERATURE-DENSITY EFFECTS

We are interested in the steady-state, gas-kinetic effects of temperature-density variations in plane-parallel media. According to equation (3.7), the relevant coupled equations are

$$\mu\frac{\partial}{\partial x}\boldsymbol{\Psi}(x,\mu)+\boldsymbol{\Psi}(x,\mu)=\frac{1}{\sqrt{\pi}}\mathbf{Q}(\mu)\int_{-\infty}^{\infty}\tilde{\mathbf{Q}}(\mu')\boldsymbol{\Psi}(x,\mu')e^{-\mu'^2}\,d\mu', \tag{4.1}$$

where $\tilde{\mathbf{Q}}(\mu)$ is the transpose of

$$\mathbf{Q}(\mu)=\begin{bmatrix}\sqrt{\frac{2}{3}}(\mu^2-\frac{1}{2}) & 1\\ \sqrt{\frac{2}{3}} & 0\end{bmatrix}, \tag{4.2}$$

and $\Psi_1(x, \mu)$ and $\Psi_2(x, \mu)$, which are sufficient to determine the temperature–density effects, are respectively the upper and lower entries in the two-vector $\boldsymbol{\Psi}(x, \mu)$. We should like to note that equation (4.1) is quite similar to the equation of transfer used in a related study [4, 5] of the scattering of polarized light.

Following Case [7] who introduced the method of normal modes in regard to one-speed neutron-transport theory, we search for elementary solutions to equation (4.1) of the form

$$\boldsymbol{\Psi}_\xi(x,\mu)=\mathbf{F}(\xi,\mu)e^{-x/\xi}, \tag{4.3}$$

where ξ and $\mathbf{F}(\xi, \mu)$ are the eigenvalues and eigenvectors to be determined. From equa-

tion (4.1), we obtain

$$(\xi - \mu)\mathbf{F}(\xi, \mu) = \frac{1}{\sqrt{\pi}}\xi\mathbf{Q}(\mu)\mathbf{M}(\xi) \tag{4.4}$$

where the normalization vector $\mathbf{M}(\xi)$ is given by

$$\mathbf{M}(\xi) = \int_{-\infty}^{\infty} \tilde{\mathbf{Q}}(\mu)\mathbf{F}(\xi, \mu)e^{-\mu^2}\,d\mu. \tag{4.5}$$

Equation (4.4) admits both discrete eigenvalues and a continuous spectrum. We consider first the discrete spectrum: $\xi = \eta_i$, $Im\ \eta_i \neq 0$, and solve equation (4.4) to obtain

$$\mathbf{F}(\eta_i, \mu) = \frac{1}{\sqrt{\pi}}\frac{\eta_i}{\eta_i - \mu}\mathbf{Q}(\mu)\mathbf{M}(\eta_i), \tag{4.6}$$

where η_i are the zeros (in the complex plane cut along the entire real axis) of the dispersion function

$$\Lambda(z) = \det \mathbf{\Lambda}(z). \tag{4.7}$$

Here the dispersion matrix is

$$\mathbf{\Lambda}(z) = \mathbf{I} + z\int_{-\infty}^{\infty} \mathbf{\Psi}(\mu)\frac{d\mu}{\mu - z}, \tag{4.8}$$

with $\mathbf{I}$ denoting the unit matrix and the characteristic matrix given by

$$\mathbf{\Psi}(\mu) = \frac{1}{\sqrt{\pi}}\tilde{\mathbf{Q}}(\mu)\mathbf{Q}(\mu)e^{-\mu^2}. \tag{4.9}$$

Further, $\mathbf{M}(\eta_i)$ is a null vector of $\mathbf{\Lambda}(\eta_i)$ such that

$$\mathbf{\Lambda}(\eta_i)\mathbf{M}(\eta_i) = \mathbf{0}. \tag{4.10}$$

The argument principle [10] may be used to show that $\Lambda(z)$ has no zeros in the finite cut plane; however, since $\Lambda(z) \sim (a/z^4) + \ldots$, for $|z|$ tending to infinity, we may deduce four 'discrete' solutions to equation (4.1). In the limit $|z| \to \infty$, we obtain from equations (4.6) and (4.10)

$$\mathbf{F}_1(\mu) = \mathbf{Q}(\mu)\begin{bmatrix}1\\0\end{bmatrix} = \sqrt{\frac{2}{3}}\begin{bmatrix}\mu^2 - \frac{1}{2}\\1\end{bmatrix}, \text{ and } \mathbf{F}_2(\mu) = \mathbf{Q}(\mu)\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}. \tag{4.11}$$

To construct the other two solutions requires a technique discussed by Case and Zweifel [8] to split the degeneracy at infinity. The resulting solutions are

$$\mathbf{\Psi}_3(x, \mu) = (\mu - x)\sqrt{\frac{2}{3}}\begin{bmatrix}\mu^2 - \frac{1}{2}\\1\end{bmatrix}, \text{ and } \mathbf{\Psi}_4(x, \mu) = (\mu - x)\begin{bmatrix}1\\0\end{bmatrix}. \tag{4.12}$$

It should be noted that equations (4.11) are solutions to equation (4.1) and to equation (4.4) in the limit $|\xi| \to \infty$; whereas, equations (4.12) are solutions only to equation (4.1).

We now consider the continuous spectrum: $\xi = \eta$, with $Im\ \eta = 0$, and the solutions to equation (4.4) are

$$\mathbf{F}(\eta, \mu) = \frac{1}{\sqrt{\pi}}\left[\eta Pv\left(\frac{1}{\eta - \mu}\right) + \lambda^*(\eta)\delta(\eta - \mu)\right]\mathbf{Q}(\mu)\mathbf{M}(\eta), \tag{4.13}$$

where $Pv(1/x)$ denotes the Cauchy principal-value distribution, and $\delta(x)$ represents the Dirac delta distribution. Pre-multiplying equation (4.13) by $\tilde{\mathbf{Q}}(\mu)\,e^{-\mu^2}$ and integrating over all μ, we find

$$[\boldsymbol{\lambda}(\eta) - \lambda^*(\eta)\boldsymbol{\Psi}(\eta)]\mathbf{M}(\eta) = \mathbf{0}, \tag{4.14}$$

where

$$\boldsymbol{\lambda}(\eta) = \mathbf{I} + \eta P \int_{-\infty}^{\infty} \boldsymbol{\Psi}(\mu)\frac{d\mu}{\mu - \eta}; \tag{4.15}$$

and hence from

$$\det\,[\boldsymbol{\lambda}(\eta) - \lambda^*(\eta)\boldsymbol{\Psi}(\eta)] = 0, \tag{4.16}$$

we obtain a quadratic equation for the function $\lambda^*(\eta)$. In general there are two solutions which we label $\lambda_1^*(\eta)$ and $\lambda_2^*(\eta)$, and thus we write the two-fold degenerate continuum solutions as

$$\mathbf{F}_\alpha(\eta, \mu) = \frac{1}{\sqrt{\pi}}\left[\eta Pv\left(\frac{1}{\eta - \mu}\right) + \lambda_\alpha^*(\eta)\delta(\eta - \mu)\right]\mathbf{Q}(\mu)\mathbf{M}_\alpha(\eta),\ \alpha = 1 \text{ or } 2,\ \eta \in (-\infty, \infty), \tag{4.17}$$

where the vectors $\mathbf{M}_\alpha(\eta)$ are to be determined by the corresponding $\lambda_\alpha^*(\eta)$, through equation (4.14).

Having established the elementary solutions, we write our general solution to equation (4.1) as

$$\boldsymbol{\Psi}(x, \mu) = \sum_{\alpha=1}^{2} A_\alpha \mathbf{F}_\alpha(\mu) + \sum_{\alpha=3}^{4} A_\alpha \boldsymbol{\Psi}_\alpha(x, \mu) + \sum_{\alpha=1}^{2} \int_{-\infty}^{\infty} A_\alpha(\eta)\mathbf{F}_\alpha(\eta, \mu)e^{-x/\eta}\,d\eta, \tag{4.18}$$

where the expansion coefficients A_α and $A_\alpha(\eta)$ are to be determined once the boundary conditions of a particular problem are specified. Although in general the integral terms in equation (4.18) may diverge for $x \neq 0$, this will not be the case when the specific problems of sections 7 and 13 are considered.

5. A FULL-RANGE EXPANSION THEOREM

To ensure that the normal modes developed in the previous sections are sufficiently general for full-range, $\mu \in (-\infty, \infty)$, boundary-value problems, we should now like to prove a basic result.

Theorem 1. *The functions* $\mathbf{F}_1(\mu)$, $\mathbf{F}_2(\mu)$, $\mathbf{F}_3(\mu) = \boldsymbol{\Psi}_3(0, \mu)$, $\mathbf{F}_4(\mu) = \boldsymbol{\Psi}_4(0, \mu)$, *and* $\mathbf{F}_\alpha(\eta, \mu)$, $\alpha = 1$ and 2, $\eta \in (-\infty, \infty)$, *form a complete basis set for the expansion of an*

arbitrary two-vector $\mathbf{I}(\mu)$, *which is Hölder continuous on any open interval of the real axis and, for sufficiently large* $|\mu|$, satisfies

$$|I_\alpha(\mu)| \exp(-|\mu|) < \infty, \quad \alpha = 1 \text{ or } 2,$$

in the sense that

$$\mathbf{I}(\mu) = \sum_{\alpha=1}^{4} A_\alpha \mathbf{F}_\alpha(\mu) + \sum_{\alpha=1}^{2} \int_{-\infty}^{\infty} A_\alpha(\eta)\mathbf{F}_\alpha(\eta, \mu)\, d\eta, \quad \mu \in (-\infty, \infty). \tag{5.1}$$

To prove the theorem, we shall construct an analytical solution to the above coupled singular integral equations. For the sake of brevity, we write

$$\hat{\mathbf{I}}(\mu) = \mathbf{I}(\mu) - \sum_{\alpha=1}^{4} A_\alpha \mathbf{F}_\alpha(\mu), \tag{5.2}$$

introduce the (2×2) matrix

$$\mathbf{G}(\eta, \mu) = [\mathbf{F}_1(\eta, \mu) \quad \mathbf{F}_2(\eta, \mu)], \tag{5.3}$$

let $\mathbf{A}(\eta)$ denote a vector with elements $A_\alpha(\eta)$, $\alpha = 1$ and 2, and thus write equation (5.1) as

$$\hat{\mathbf{I}}(\mu) = \int_{-\infty}^{\infty} \mathbf{G}(\eta, \mu)\mathbf{A}(\eta)\, d\eta, \quad \mu \in (-\infty, \infty). \tag{5.4}$$

Pre-multiplying equation (4.13) by $\tilde{\mathbf{Q}}(\mu)\, e^{-\mu^2}$ and invoking equation (4.14), we obtain

$$\tilde{\mathbf{Q}}(\mu)\mathbf{G}(\eta, \mu)\, e^{-\mu^2} = \left[\eta Pv\left(\frac{1}{\eta - \mu}\right)\mathbf{\Psi}(\mu) + \delta(\eta - \mu)\boldsymbol{\lambda}(\eta)\right]\mathbf{V}(\eta), \tag{5.5}$$

where

$$\mathbf{V}(\eta) = [\mathbf{M}_1(\eta) \quad \mathbf{M}_2(\eta)] \tag{5.6}$$

is the (2×2) normalization matrix. We now pre-multiply equation (5.4) by $\tilde{\mathbf{Q}}(\mu)\, e^{-\mu^2}$, make use of equation (5.5), and integrate the δ term to obtain

$$\tilde{\mathbf{Q}}(\mu)\hat{\mathbf{I}}(\mu)e^{-\mu^2} = \boldsymbol{\lambda}(\mu)\mathbf{B}(\mu) + \mathbf{\Psi}(\mu)P\int_{-\infty}^{\infty} \eta\mathbf{B}(\eta)\frac{d\eta}{\eta - \mu}, \tag{5.7}$$

where $\mathbf{B}(\eta) = \mathbf{V}(\eta)\mathbf{A}(\eta)$. Equation (5.7) may now be solved explicitly by using the theory of Muskhelishvili [21]. To convert equation (5.7) to a special form of a matrix Riemann problem, we introduce the sectionally holomorphic matrix

$$\mathbf{N}(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \eta\mathbf{B}(\eta)\frac{d\eta}{\eta - z}. \tag{5.8}$$

The boundary values of $\mathbf{N}(z)$ as z approaches the real line from above (+) and below

(–) follow from the Plemelj formulae[21]:

$$\pi i[\mathbf{N}^+(\mu)+\mathbf{N}^-(\mu)] = P\int_{-\infty}^{\infty} \eta\mathbf{B}(\eta)\frac{d\eta}{\eta-\mu} \tag{5.9a}$$

and

$$\mathbf{N}^+(\mu)-\mathbf{N}^-(\mu) = \mu\mathbf{B}(\mu). \tag{5.9b}$$

In a similar manner, the boundary values of the dispersion matrix follow from equation (4.8):

$$\mathbf{\Lambda}^+(\mu)+\mathbf{\Lambda}^-(\mu) = 2\boldsymbol{\lambda}(\mu), \tag{5.10a}$$

and

$$\mathbf{\Lambda}^+(\mu)-\mathbf{\Lambda}^-(\mu) = 2\pi i\mu\mathbf{\Psi}(\mu). \tag{5.10b}$$

Equations (5.9) and (5.10) may now be used in equation (5.7) to yield

$$\mu\tilde{\mathbf{Q}}(\mu)\hat{\mathbf{I}}(\mu)e^{-\mu^2} = \mathbf{\Lambda}^+(\mu)\mathbf{N}^+(\mu)-\mathbf{\Lambda}^-(\mu)\mathbf{N}^-(\mu), \tag{5.11}$$

which can be solved to give

$$\mathbf{N}(z) = \mathbf{\Lambda}^{-1}(z)\frac{1}{2\pi i}\int_{-\infty}^{\infty} \mu\tilde{\mathbf{Q}}(\mu)\hat{\mathbf{I}}(\mu)e^{-\mu^2}\frac{d\mu}{\mu-z}. \tag{5.12}$$

We note that for large $|z|$,

$$\mathbf{\Lambda}(z) \sim -\frac{1}{z^2}\begin{bmatrix} \frac{7}{6} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & \frac{1}{2} \end{bmatrix}, \text{ as } |z|\to\infty, \tag{5.13}$$

and

$$\mathbf{\Lambda}^{-1}(z) \sim -\frac{12}{5}z^2\begin{bmatrix} \frac{1}{2} & \frac{-1}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} & \frac{7}{6} \end{bmatrix}, \text{ as } |z|\to\infty, \tag{5.14}$$

and therefore if the $\mathbf{N}(z)$ as given by equation (5.12) is to vanish when $|z|$ tends to infinity, as equation (5.8) prescribes, we must impose on the vector $\hat{\mathbf{I}}(\mu)$ the four constraints

$$\int_{-\infty}^{\infty} \mu^\alpha\tilde{\mathbf{Q}}(\mu)\hat{\mathbf{I}}(\mu)e^{-\mu^2}\,d\mu = \mathbf{0}, \quad \alpha = 1 \text{ and } 2. \tag{5.15}$$

Recalling equation (5.2) for $\hat{\mathbf{I}}(\mu)$, we observe that equation (5.15) will be inherently

satisfied if we specify the expansion coefficients A_α, $\alpha = 1, 2, 3$, and 4, to be

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = \frac{12}{5\sqrt{6\pi}} \begin{bmatrix} -\frac{3}{2}I_{12} + I_{14} + I_{22} \\ \frac{8}{\sqrt{6}}I_{12} - \sqrt{\frac{2}{3}}I_{14} - \sqrt{\frac{2}{3}}I_{22} \\ -\frac{3}{2}I_{11} + I_{13} + I_{21} \\ \frac{8}{\sqrt{6}}I_{11} - \sqrt{\frac{2}{3}}I_{13} - \sqrt{\frac{2}{3}}I_{21} \end{bmatrix}, \tag{5.16}$$

where

$$I_{\alpha\beta} \triangleq \int_{-\infty}^{\infty} \mu^\beta I_\alpha(\mu) e^{-\mu^2} \, d\mu, \quad \alpha = 1 \text{ or } 2; \beta = 1, 2, 3 \text{ or } 4, \tag{5.17}$$

and $I_1(\mu)$ and $I_2(\mu)$ are respectively the upper and lower entries of $\mathbf{I}(\mu)$. Theorem 1 is therefore established.

Although we could pursue equations (5.9b) and (5.12) to obtain explicit results for the continuum coefficients $A_\alpha(\eta)$, $\alpha = 1$ or 2, we prefer to summarize the final expressions in terms of the formalism of the full-range orthogonality relations given in the next section.

6. ORTHOGONALITY RELATIONS AND EXPLICIT SOLUTIONS

We should first like to state the general orthogonality relation relevant to all solutions including the special distributions, $\mathbf{F}(\xi, \mu)$, of the separated equation (4.4).

Theorem 2. *All eigenvectors* $\mathbf{F}(\xi, \mu)$ *which are solutions of equation* (4.4) *are orthogonal on the full range,* $\mu \in (-\infty, \infty)$, *in the sense that*

$$\int_{-\infty}^{\infty} \tilde{\mathbf{F}}(\xi', \mu)\mathbf{F}(\xi, \mu) e^{-\mu^2} \mu \, d\mu = 0, \quad \xi' \neq \xi. \tag{6.1}$$

To prove the theorem, equation (4.4) is first pre-multiplied by $\tilde{\mathbf{F}}(\xi', \mu)e^{-\mu^2}/\xi$, the transpose of equation (4.4) with ξ changed to ξ' is post-multiplied by $\mathbf{F}(\xi, \mu)e^{-\mu^2}/\xi'$, and the two resulting equations are then integrated over all μ and subtracted one from the other to yield

$$\left(\frac{1}{\xi} - \frac{1}{\xi'}\right) \int_{-\infty}^{\infty} \tilde{\mathbf{F}}(\xi', \mu)\mathbf{F}(\xi, \mu) e^{-\mu^2} \mu \, d\mu = 0, \tag{6.2}$$

which establishes equation (6.1). Though equation (6.2) is a general statement of full-range orthogonality, it is clear that several additional relations are required here. First of all, since $\mathbf{F}_1(\mu)$ and $\mathbf{F}_2(\mu)$ are both associated with $\xi \to \infty$, equation (6.2) does not ensure that they will be mutually orthogonal in the sense of equation (6.1). In addition, the vectors $\mathbf{F}_3(\mu)$ and $\mathbf{F}_4(\mu)$, being derived from the solutions of equation (4.1), rather than equation (4.4), are not included in Theorem 2. However, it can be easily shown that $\mathbf{F}_1(\mu)$ and $\mathbf{F}_2(\mu)$ are mutually orthogonal, and, in fact, self-orthogonal; the same is true for $\mathbf{F}_3(\mu)$ and $\mathbf{F}_4(\mu)$. In addition, $\mathbf{F}_3(\mu)$ and $\mathbf{F}_4(\mu)$ are orthogonal to the continuum

solutions $\mathbf{F}_1(\eta, \mu)$ and $\mathbf{F}_2(\eta, \mu)$. We note that $\mathbf{F}_1(\mu)$ and $\mathbf{F}_2(\mu)$ are not orthogonal to $\mathbf{F}_3(\mu)$ and $\mathbf{F}_4(\mu)$; however, suitably defined adjoint vectors for these special cases can be developed by employing a Schmidt-type procedure.

Considering first the normalization integrals related to the solutions given by equation (4.13), we find

$$\int_{-\infty}^{\infty} \tilde{\mathbf{F}}_\alpha(\eta', \mu)\mathbf{F}_\beta(\eta, \mu)e^{-\mu^2}\,\mu\,d\mu = S(\eta)\delta(\eta - \eta')\delta_{\alpha,\beta}\,;\ \alpha,\beta = 1, 2, \tag{6.3}$$

where

$$S(\eta) = \frac{\eta}{\sqrt{\pi}}[\pi^2\eta^2 + \lambda_\alpha^*(\eta)\lambda_\beta^*(\eta)]\tilde{\mathbf{M}}_\alpha(\eta)\mathbf{\Psi}(\eta)\mathbf{M}_\beta(\eta). \tag{6.4}$$

The Kronecker $\delta_{\alpha,\beta}$ appearing in equation (6.3) should be noted since it ensures that the degenerate continuum solutions given by equation (4.13) are orthogonal even for $\eta' = \eta$. To establish equation (6.3) requires the use of the Poincaré-Bertrand formula [21] and

$$[\lambda_\alpha^*(\eta) - \lambda_\beta^*(\eta)]\tilde{\mathbf{M}}_\alpha(\eta)\mathbf{\Psi}(\eta)\mathbf{M}_\beta(\eta) = 0, \tag{6.5}$$

a relation which can be deduced from equation (4.14).

Though the representations of the two continuum solutions given by equation (4.13) were convenient for proving the full-range expansion theorem, we choose to make use of more explicit forms for actual applications. We note that equation (4.16) is quadratic in $\lambda^*(\eta)$, and thus the two solutions will in general involve radicals. To avoid the cumbersome nature of the ensuing solutions, we prefer the linear combinations

$$\mathbf{\Phi}_\alpha(\eta, \mu) = T_{\alpha 1}(\eta)\mathbf{F}_1(\eta, \mu) + T_{\alpha 2}(\eta)\mathbf{F}_2(\eta, \mu), \quad \alpha = 1 \text{ and } 2, \tag{6.6}$$

which, for judicious choices of $T_{\alpha\beta}(\eta)$, enable us to deduce the more tractable solutions

$$\mathbf{\Phi}_1(\eta, \mu) = \begin{bmatrix} \frac{1}{\sqrt{\pi}}\eta\left(\mu^2 - \frac{1}{2}\right)Pv\left(\frac{1}{\eta - \mu}\right)e^{-\eta^2} + \lambda_1(\eta)\delta(\eta - \mu) \\ \frac{1}{\sqrt{\pi}}\eta Pv\left(\frac{1}{\eta - \mu}\right)e^{-\eta^2} + \left[\frac{1}{2} + \lambda_0(\eta)\right]\delta(\eta - \mu) \end{bmatrix} \tag{6.7a}$$

and

$$\mathbf{\Phi}_2(\eta, \mu) = \begin{bmatrix} \frac{1}{\sqrt{\pi}}\eta Pv\left(\frac{1}{\eta - \mu}\right)e^{-\eta^2} + \lambda_0(\eta)\delta(\eta - \mu) \\ \frac{1}{2}\delta(\eta - \mu) \end{bmatrix}, \tag{6.7b}$$

where

$$\lambda_0(\eta) = 1 + \frac{1}{\sqrt{\pi}}\eta P\int_{-\infty}^{\infty} e^{-\mu^2}\frac{d\mu}{\mu - \eta}, \tag{6.8a}$$

or

$$\lambda_0(\eta) = 1 - 2\eta e^{-\eta^2} \int_0^\eta e^{\mu^2} d\mu, \tag{6.8b}$$

and

$$\lambda_1(\eta) = \tfrac{1}{2} + (\eta^2 - \tfrac{1}{2})\lambda_0(\eta). \tag{6.9}$$

We note that equations (6.7) are not mutually orthogonal for $\eta' = \eta$; however, a Schmidt-type procedure may be used here as well. Since our final adjoint vectors follow in a manner analogous to that reported by Siewert and Zweifel [26], we shall simply summarize our conclusions below. For the case of the degenerate continuum modes, we find that the procedure discussed in reference [26] can be used to establish the required adjoint vectors. To unify our notation, we also define

$$\boldsymbol{\Phi}_\alpha(\mu) \stackrel{\Delta}{=} \mathbf{F}_\alpha(\mu), \quad \alpha = 1, 2, 3 \text{ and } 4. \tag{6.10}$$

The *orthonormal* full-range adjoint set is given by:

$$\mathbf{X}_1(\mu) = \frac{1}{5\sqrt{\pi}}[6\boldsymbol{\Phi}_3(\mu) - 2\sqrt{6}\boldsymbol{\Phi}_4(\mu)], \tag{6.11a}$$

$$\mathbf{X}_2(\mu) = \frac{1}{5\sqrt{\pi}}[-2\sqrt{6}\boldsymbol{\Phi}_3(\mu) + 14\boldsymbol{\Phi}_4(\mu)], \tag{6.11b}$$

$$\mathbf{X}_3(\mu) = \frac{1}{5\sqrt{\pi}}[6\boldsymbol{\Phi}_1(\mu) - 2\sqrt{6}\boldsymbol{\Phi}_2(\mu)], \tag{6.11c}$$

$$\mathbf{X}_4(\mu) = \frac{1}{5\sqrt{\pi}}[-2\sqrt{6}\boldsymbol{\Phi}_1(\mu) + 14\boldsymbol{\Phi}_2(\mu)], \tag{6.11d}$$

$$\mathbf{X}_1(\eta, \mu) = \frac{1}{N(\eta)}[N_{22}(\eta)\boldsymbol{\Phi}_1(\eta, \mu) - N_{12}(\eta)\boldsymbol{\Phi}_2(\eta, \mu)], \tag{6.11e}$$

and

$$\mathbf{X}_2(\eta, \mu) = \frac{1}{N(\eta)}[N_{11}(\eta)\boldsymbol{\Phi}_2(\eta, \mu) - N_{12}(\eta)\boldsymbol{\Phi}_1(\eta, \mu)], \tag{6.11f}$$

where

$$N_{11}(\eta) = [\lambda_0(\eta) + \tfrac{1}{2}]^2 + \lambda_1^2(\eta) + \pi\eta^2[(\eta^2 - \tfrac{1}{2})^2 + 1]\, e^{-2\eta^2},$$

$$N_{12}(\eta) = \lambda_0(\eta)\lambda_1(\eta) + \tfrac{1}{2}\lambda_0(\eta) + \tfrac{1}{4} + \pi\eta^2(\eta^2 - \tfrac{1}{2})\, e^{-2\eta^2},$$

$$N_{22}(\eta) = \lambda_0^2(\eta) + \tfrac{1}{4} + \pi\eta^2\, e^{-2\eta^2},$$

and

$$N(\eta) = \tfrac{9}{4}\eta\, e^{-\eta^2}\Lambda^+(\eta)\Lambda^-(\eta). \tag{6.12}$$

The required orthogonality relations among the full-range basis and adjoint sets are:

$$\int_{-\infty}^{\infty} \tilde{\mathbf{X}}_\alpha(\mu)\boldsymbol{\Phi}_\beta(\mu)\,\mathrm{e}^{-\mu^2}\,\mu\,\mathrm{d}\mu = \delta_{\alpha,\beta};\ \alpha,\beta = 1, 2, 3, \text{or } 4, \tag{6.13a}$$

$$\int_{-\infty}^{\infty} \tilde{\mathbf{X}}_\alpha(\mu)\boldsymbol{\Phi}_\beta(\eta, \mu)\,\mathrm{e}^{-\mu^2}\,\mu\,\mathrm{d}\mu = 0;\ \alpha = 1, 2, 3, \text{or } 4,\ \beta = 1 \text{ or } 2, \tag{6.13b}$$

$$\int_{-\infty}^{\infty} \tilde{\mathbf{X}}_\alpha(\eta', \mu)\boldsymbol{\Phi}_\beta(\eta, \mu)\,\mathrm{e}^{-\mu^2}\,\mu\,\mathrm{d}\mu = \delta(\eta - \eta')\delta_{\alpha,\beta};\ \alpha,\beta = 1 \text{ or } 2, \tag{6.13c}$$

$$\int_{-\infty}^{\infty} \tilde{\mathbf{X}}_\alpha(\eta', \mu)\boldsymbol{\Phi}_\beta(\mu)\,\mathrm{e}^{-\mu^2}\,\mu\,\mathrm{d}\mu = 0;\ \alpha = 1 \text{ or } 2,\ \beta = 1, 2, 3, \text{or } 4. \tag{6.13d}$$

With the formalism thus established, we note that all expansion coefficients in equations of the form

$$\mathbf{I}(\mu) = \sum_{\alpha=1}^{4} A_\alpha \boldsymbol{\Phi}_\alpha(\mu) + \sum_{\alpha=1}^{2} \int_{-\infty}^{\infty} A_\alpha(\eta)\boldsymbol{\Phi}_\alpha(\eta, \mu)\,\mathrm{d}\eta,\ \mu \in (-\infty, \infty), \tag{6.14}$$

may be expressed immediately in terms of inner products:

$$A_\alpha = \int_{-\infty}^{\infty} \tilde{\mathbf{X}}_\alpha(\mu)\mathbf{I}(\mu)\,\mathrm{e}^{-\mu^2}\,\mu\,\mathrm{d}\mu,\ \alpha = 1, 2, 3 \text{ and } 4, \tag{6.15}$$

and

$$A_\alpha(\eta) = \int_{-\infty}^{\infty} \tilde{\mathbf{X}}_\alpha(\eta, \mu)\mathbf{I}(\mu)\,\mathrm{e}^{-\mu^2}\,\mu\,\mathrm{d}\mu,\ \alpha = 1 \text{ and } 2. \tag{6.16}$$

7. THE INFINITE-MEDIUM GREEN'S FUNCTION

In order to illustrate the use of the elementary solutions of equation (4.1) and the relevant orthogonality relations, we should now like to develop the infinite-medium Green's function. Here we seek a solution to

$$\mu\frac{\partial}{\partial x}\boldsymbol{\Psi}(x, \mu) + \boldsymbol{\Psi}(x, \mu) = \frac{1}{\sqrt{\pi}}\mathbf{Q}(\mu)\int_{-\infty}^{\infty} \tilde{\mathbf{Q}}(\mu')\boldsymbol{\Psi}(x, \mu')\,\mathrm{e}^{-\mu'^2}\,\mathrm{d}\mu' + \mathbf{S}(x_0, \mu_1, \mu_2; x, \mu), \tag{7.1}$$

where

$$\mathbf{S}(x_0, \mu_1, \mu_2; x, \mu) = \delta(x - x_0)\begin{bmatrix}\rho_1\,\delta(\mu - \mu_1)\\ \rho_2\,\delta(\mu - \mu_2)\end{bmatrix}. \tag{7.2}$$

Clearly, since the kinetic equation conserves particles, kinetic energy, and momentum, there will exist no bounded (at infinity) solution to equation (7.1); however, the Green's function we develop may be used in the classical manner to construct particular solutions to equation (7.1) for arbitrary inhomogeneous source terms for semi-infinite or finite media. As discussed by Case and Zweifel [8], we neglect the inhomogeneous term

in equation (7.1) and require the solution to the resulting homogeneous equation to satisfy the 'jump' boundary condition

$$\mu[\boldsymbol{\Psi}(x_0, \mu_1, \mu_2; x_0^+, \mu) - \boldsymbol{\Psi}(x_0, \mu_1, \mu_2; x_0^-, \mu)] = \begin{bmatrix} \rho_1\, \delta(\mu - \mu_1) \\ \rho_2\, \delta(\mu - \mu_2) \end{bmatrix}, \quad \mu \in (-\infty, \infty), \tag{7.3}$$

where the argument list has been extended to include the parameters x_0, μ_1, and μ_2. We therefore write the desired solution as

$$\boldsymbol{\Psi}(x_0, \mu_1, \mu_2; x, \mu) = \sum_{\alpha=1}^{2} A_\alpha \boldsymbol{\Phi}_\alpha(\mu) + \sum_{\alpha=1}^{2} \int_0^\infty A_\alpha(\eta)\boldsymbol{\Phi}_\alpha(\eta, \mu)\, e^{-(x-x_0)/\eta}\, d\eta, \tag{7.4a}$$

for $x > x_0$,

and

$$\boldsymbol{\Psi}(x_0, \mu_1, \mu_2; x, \mu) = -\sum_{\alpha=3}^{4} A_\alpha \boldsymbol{\Psi}_\alpha(x - x_0, \mu) - \sum_{\alpha=1}^{2} \int_{-\infty}^{0} A_\alpha(\eta)\boldsymbol{\Phi}_\alpha(\eta, \mu)\, e^{-(x-x_0)/\eta}\, d\eta$$

for $x < x_0$. (7.4b)

Substitution of equations (7.4) into equation (7.3) yields the full-range expansion

$$\begin{bmatrix} \rho_1\, \delta(\mu - \mu_1) \\ \rho_2\, \delta(\mu - \mu_2) \end{bmatrix} = \mu\left\{\sum_{\alpha=1}^{4} A_\alpha \boldsymbol{\Phi}_\alpha(\mu) + \sum_{\alpha=1}^{2} \int_{-\infty}^{\infty} A_\alpha(\eta)\boldsymbol{\Phi}_\alpha(\eta, \mu)\, d\eta\right\}, \quad \mu \in (-\infty, \infty). \tag{7.5}$$

Though the left-hand side of equation (7.5) certainly is not a Hölder function, Case and Zweifel[8] have concluded that expansion theorems similar to our Theorem 1 remain valid formally even for this type of delta distribution. We therefore pre-multiply equation (7.5) by $\tilde{\mathbf{X}}_\alpha(\mu)\, e^{-\mu^2}$, $\alpha = 1, 2, 3$, or 4 and $\tilde{\mathbf{X}}_\alpha(\eta', \mu)\, e^{-\mu^2}$, $\alpha = 1$ or 2, and integrate over all μ to find, after invoking equation (6.13),

$$A_\alpha = \rho_1 X_{\alpha 1}(\mu_1)\, e^{-\mu_1^2} + \rho_2 X_{\alpha 2}(\mu_2)\, e^{-\mu_2^2}, \quad \alpha = 1, 2, 3, \text{ and } 4, \tag{7.6a}$$

and

$$A_\alpha(\eta) = \rho_1 X_{\alpha 1}(\eta, \mu_1)\, e^{-\mu_1^2} + \rho_2 X_{\alpha 2}(\eta, \mu_2)\, e^{-\mu_2^2}, \quad \alpha = 1 \text{ and } 2, \tag{7.6b}$$

where the subscripts 1 and 2 are used to denote the upper and lower elements of the $\mathbf{X}$ vectors. Since all expansion coefficients required in equations (7.4) are given by equations (7.6), the infinite-medium Green's function is established.

8. A HALF-RANGE EXPANSION THEOREM

Having developed in sections 5 and 6 the necessary completeness and orthogonality properties of our normal modes, we should now like to discuss the analysis required for the considerably more interesting problems defined by half-range, $\mu \in (0, \infty)$, boundary conditions. The following theorem states the very important half-range expansion properties basic to a certain subset of our derived elementary solutions.

Theorem 3. *The functions* $\mathbf{F}_1(\mu)$, $\mathbf{F}_2(\mu)$ *and* $\mathbf{F}_\alpha(\eta, \mu)$, $\alpha = 1$ *and* 2, $\eta \in (0, \infty)$, *form a complete basis set for the expansion of an arbitrary two-vector* $\mathbf{I}(\mu)$ *which is Hölder*

continuous on any open interval of the positive real axis and, for sufficiently large $|\mu|$, satisfies

$$|\mathrm{I}_\alpha(\mu)| \exp(-|\mu|) < \infty, \ \alpha = 1 \text{ or } 2,$$

in the sense that

$$\mathbf{I}(\mu) = \sum_{\alpha=1}^{2} A_\alpha \mathbf{F}_\alpha(\mu) + \sum_{\alpha=1}^{2} \int_0^\infty A_\alpha(\eta)\mathbf{F}_\alpha(\eta, \mu)\, \mathrm{d}\eta, \ \mu \in (0, \infty). \tag{8.1}$$

To prove this theorem, we premultiply equation (8.1) by $\mathrm{e}^{-\mu^2}\, \tilde{\mathbf{Q}}(\mu)$, integrate the δ term and use equations (4.13) and (4.14) to obtain

$$\tilde{\mathbf{Q}}(\mu)\hat{\mathbf{I}}(\mu)\, \mathrm{e}^{-\mu^2} = \boldsymbol{\lambda}(\mu)\mathbf{B}(\mu) + \boldsymbol{\Psi}(\mu) P \int_0^\infty \eta \mathbf{B}(\eta) \frac{\mathrm{d}\eta}{\eta - \mu}, \ \mu \in (0, \infty), \tag{8.2}$$

where

$$\hat{\mathbf{I}}(\mu) = \mathbf{I}(\mu) - \sum_{\alpha=1}^{2} A_\alpha \mathbf{F}_\alpha(\mu) \tag{8.3}$$

and

$$\mathbf{B}(\eta) = \mathbf{V}(\eta)\mathbf{A}(\eta). \tag{8.4}$$

In addition, $\mathbf{V}(\eta)$ is given by equation (5.6) and the unknown $\mathbf{A}(\eta)$ has elements $A_1(\eta)$ and $A_2(\eta)$. In a manner similar to that used to prove Theorem 1, we now introduce

$$\mathbf{N}(z) = \frac{1}{2\pi i} \int_0^\infty \eta \mathbf{B}(\eta) \frac{\mathrm{d}\eta}{\eta - z}. \tag{8.5}$$

The $\mathbf{N}$ matrix is clearly analytic in the complex plane cut along the positive real axis. Further, the Plemelj formulae[21] can be used, with equation (8.5), to show that the boundary values of $\mathbf{N}(z)$ satisfy

$$\pi i[\mathbf{N}^+(\mu) + \mathbf{N}^-(\mu)] = P \int_0^\infty \eta \mathbf{B}(\eta) \frac{\mathrm{d}\eta}{\eta - \mu} \tag{8.6a}$$

and

$$\mathbf{N}^+(\mu) - \mathbf{N}^-(\mu) = \mu \mathbf{B}(\mu). \tag{8.6b}$$

Equations (8.6) can now be used, along with equations (5.10), to express equation (8.2) in the form

$$\mu \tilde{\mathbf{Q}}(\mu)\hat{\mathbf{I}}(\mu)\, \mathrm{e}^{-\mu^2} = \boldsymbol{\Lambda}^+(\mu)\mathbf{N}^+(\mu) - \boldsymbol{\Lambda}^-(\mu)\mathbf{N}^-(\mu), \ \mu \in (0, \infty). \tag{8.7}$$

If we now let $\tilde{\mathbf{X}}(z)$ denote a canonical (non-singular in the finite plane) solution to the homogeneous Riemann problem defined by

$$\tilde{\mathbf{X}}^+(\mu) = \mathbf{G}(\mu)\tilde{\mathbf{X}}^-(\mu), \ \mu \in (0, \infty), \tag{8.8a}$$

where

$$\mathbf{G}(\mu) = \mathbf{\Lambda}^{+}(\mu)[\mathbf{\Lambda}^{-}(\mu)]^{-1}, \tag{8.8b}$$

then equation (8.7) can be solved immediately to yield

$$\mathbf{N}(z) = \mathbf{X}^{-1}(z)\left[\frac{1}{2\pi i}\int_0^\infty \mathbf{\Gamma}(\mu)\tilde{\mathbf{Q}}(\mu)\hat{\mathbf{I}}(\mu)\,\mathrm{e}^{-\mu^2}\frac{\mathrm{d}\mu}{\mu - z} + \mathbf{P}(z)\right]. \tag{8.9}$$

Here, $\mathbf{P}(z)$ is a matrix of polynomials, and

$$\mathbf{\Gamma}(\mu) = \mu\mathbf{X}^{+}(\mu)[\mathbf{\Lambda}^{+}(\mu)]^{-1}. \tag{8.10}$$

Since the $\mathbf{G}$ matrix given by equation (8.8b) is continuous for $\mu \in [0, \infty)$, $\mathbf{G}(0) = \mathbf{I}$ and $\mathbf{G}(\mu) \to \mathbf{I}$ as $\mu \to \infty$, the analysis of Mandžavidze and Hvedelidze [20] can be used, after an elementary transformation of variables, to ensure the existence of a canonical solution to the Riemann problem defined by equation (8.8a). In section 9 we argue that the partial indices κ_1 and κ_2 associated with our canonical solution $\tilde{\mathbf{X}}(z)$ are

$$\kappa_1 = \kappa_2 = 1, \tag{8.11}$$

and thus if we allow our canonical matrix $\tilde{\mathbf{X}}(z)$ to be of normal form at infinity [21], we may write

$$\lim_{z\to\infty} z\,\mathbf{X}(z) = \mathbf{\Delta}, \tag{8.12}$$

where $\mathbf{\Delta}$ is nonsingular and bounded.

From the defining equation (8.5), we observe that $z\,\mathbf{N}(z)$ must be bounded as $|z| \to \infty$, and thus from equation (8.9) we conclude that

$$\lim_{|z|\to\infty} z\mathbf{\Delta}^{-1}\left[\frac{-1}{2\pi i}\int_0^\infty \mathbf{\Gamma}(\mu)\tilde{\mathbf{Q}}(\mu)\hat{\mathbf{I}}(\mu)\,\mathrm{e}^{-\mu^2}\,\mathrm{d}\mu + z\mathbf{P}(z)\right] < \infty; \tag{8.13}$$

we must therefore set $\mathbf{P}(z) = \mathbf{0}$ and, in addition, require that

$$\int_0^\infty \mathbf{\Gamma}(\mu)\tilde{\mathbf{Q}}(\mu)\hat{\mathbf{I}}(\mu)\,\mathrm{e}^{-\mu^2}\,\mathrm{d}\mu = \mathbf{0}. \tag{8.14}$$

Equation (8.14) is, of course, not satisfied by all $\hat{\mathbf{I}}(\mu)$, but recalling equation (8.3), we conclude that choosing the discrete expansion coefficients to be solutions of

$$\int_0^\infty \mathbf{\Gamma}(\mu)\mathbf{\Psi}(\mu)\,\mathrm{d}\mu\begin{bmatrix}A_1\\A_2\end{bmatrix} = \frac{1}{\sqrt{\pi}}\int_0^\infty \mathbf{\Gamma}(\mu)\tilde{\mathbf{Q}}(\mu)\mathbf{I}(\mu)\,\mathrm{e}^{-\mu^2}\,\mathrm{d}\mu \tag{8.15}$$

renders equation (8.1) valid for all appropriate $\mathbf{I}(\mu)$. The matrix in equation (8.15) whose inverse is required to obtain A_1 and A_2 can be shown to be non-singular by making use of a Cauchy integral representation of $\mathbf{X}(z)$. The theorem is therefore proved.

Though, as for the full-range case, we could pursue this completeness proof to construct the continuum expansion coefficients $\mathbf{A}(\eta)$, $\eta \in (0, \infty)$, we find it more convenient to express the final results in terms of half-range orthogonality relations.

9. A PROOF REGARDING THE PARTIAL INDICES OF THE RIEMANN PROBLEM

The proof of the half-range expansion theorem given in section 8 is based on the proposition that the partial indices κ_1 and κ_2 are both non-negative. In fact, equation (8.13) is valid only if the partial indices are given by $\kappa_1 = \kappa_2 = 1$. In this section we develop the required proof that $\kappa_1 = \kappa_2 = 1$.

We consider then the homogeneous matrix Riemann problem defined by

$$\boldsymbol{\Phi}^+(\mu) = \mathbf{G}(\mu)\boldsymbol{\Phi}^-(\mu), \ \mu \in [0, \infty), \tag{9.1}$$

where

$$\mathbf{G}(\mu) = \boldsymbol{\Lambda}^+(\mu)[\boldsymbol{\Lambda}^-(\mu)]^{-1}, \ \mu \in [0, \infty). \tag{9.2}$$

Here we seek a matrix $\boldsymbol{\Phi}(z)$ analytic in the plane cut along the positive real axis, non-singular in the finite plane, and with boundary values $\boldsymbol{\Phi}^{\pm}(\mu)$ which satisfy equation (9.1).

Since $\mathbf{G}(0) = \mathbf{I}$ and $\mathbf{G}(\mu) \to \mathbf{I}$ in the limit as $\mu \to \infty$, we can define $\mathbf{G}(\mu) = \mathbf{I}$ on the entire negative real axis and thus consider equation (9.1) for $\mu \in (-\infty, \infty)$. To make use of the results developed by Mandžavidze and Hvedelidze[20], valid for closed contours, we make the change of variables

$$\zeta = \frac{i - z}{i + z} \tag{9.3}$$

which maps the upper half of the z plane into the interior of the unit circle in the ζ-plane. We note that the positive (negative) real axis maps into $|\zeta| = 1$, $Im\ \zeta > (<)\ 0$. The existence of a solution to the Riemann problem in the ζ plane follows from the theory of Mandžavidze and Hvedelidze[20], since the resulting $\mathbf{G}$ matrix is continuous on the unit circle, and $\boldsymbol{\Phi}(z)$, the canonical solution in the z-plane, is the image of the solution in the ζ plane postmultiplied by an appropriate matrix of rational functions.

It can be demonstrated[5] that the $\boldsymbol{\Lambda}$ matrix can be factored as

$$\boldsymbol{\Lambda}(z) = \boldsymbol{\Phi}(z)\mathcal{P}(z)\tilde{\boldsymbol{\Phi}}(-z) \tag{9.4}$$

where $\boldsymbol{\Phi}(z)$ is any canonical solution (of ordered normal form at infinity) to equation (9.1) and $\mathcal{P}(z)$ is a matrix of polynomials, which depends on the particular choice for $\boldsymbol{\Phi}(z)$.

The fact that $\overline{\mathbf{G}(\mu)} = [\mathbf{G}(\mu)]^{-1}$, where the bar indicates the complex conjugate, enables us to extend the results of Siewert and Burniston's[25] Theorem II to the Riemann problem defined by equation (9.1):

Theorem 4. *There exists at least one canonical matrix* $\boldsymbol{\Phi}_1(z)$ *of ordered normal form at infinity for the Riemann problem defined by equation* (9.1) *such that* $\overline{\boldsymbol{\Phi}_1(\bar{z})} = \boldsymbol{\Phi}_1(z)$.

Since the proof of Theorem 4 follows very closely one previously reported [25], it will not be given here.

If we use $\boldsymbol{\Phi}_1(z)$ in the factorization of $\boldsymbol{\Lambda}(z)$, the resulting polynomial matrix $\mathcal{P}(z)$ is such that $\mathcal{P}(z) = \tilde{\mathcal{P}}(-z)$ and $\mathcal{P}(z) = \overline{\mathcal{P}(\bar{z})}$, since $\boldsymbol{\Lambda}(z) = \tilde{\boldsymbol{\Lambda}}(z) = \boldsymbol{\Lambda}(-z)$ and $\boldsymbol{\Lambda}(z) = \overline{\boldsymbol{\Lambda}(\bar{z})}$.

By definition [21], a canonical solution of ordered normal form at infinity is such that

$$\lim_{|z|\to\infty} \boldsymbol{\Phi}_1(z)\begin{bmatrix} z^{\kappa_1} & 0 \\ 0 & z^{\kappa_2} \end{bmatrix} = \mathbf{K}, \quad \det \mathbf{K} \neq 0, \tag{9.5}$$

where κ_1 and $\kappa_2 \geq \kappa_1$ are the partial indices. Furthermore the sum of κ_1 and κ_2 must yield the total index κ, which in the manner of Muskelishvili [21] can be computed directly once the **G** matrix and the appropriate contour are specified. For this problem, we find

$$\kappa_1 + \kappa_2 = \kappa = 2. \tag{9.6}$$

If we now evaluate equation (9.4), for $\boldsymbol{\Phi}(z) = \boldsymbol{\Phi}_1(z)$ at $z = 0$, we obtain

$$\mathcal{P}(0) = \boldsymbol{\Phi}_1^{-1}(0)\tilde{\boldsymbol{\Phi}}_1^{-1}(0), \tag{9.7}$$

and since $\boldsymbol{\Phi}_1(0)$ is real (recall that $\boldsymbol{\Phi}_1(z) = \overline{\boldsymbol{\Phi}_1(\bar{z})}$), we conclude from equation (9.7) that $\mathcal{P}_{11}(0) \neq 0$ and $\mathcal{P}_{22}(0) \neq 0$. Again from equation (9.4), for $\boldsymbol{\Phi}(z) = \boldsymbol{\Phi}_1(z)$, we can write, after using equation (5.13),

$$\mathcal{P}(z) \to -\frac{1}{z^2}\begin{bmatrix} z^{\kappa_1} & 0 \\ 0 & z^{\kappa_2} \end{bmatrix} \mathbf{K}^{-1}\begin{bmatrix} \frac{7}{6} & \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{6} & \frac{1}{2} \end{bmatrix}\tilde{\mathbf{K}}^{-1}\begin{bmatrix} (-z)^{\kappa_1} & 0 \\ 0 & (-z)^{\kappa_2} \end{bmatrix}, \tag{9.8}$$

$$\text{as } |z| \to \infty,$$

from which it follows, since **K** is real, that

$$\kappa_1 = \kappa_2 = 1. \tag{9.9}$$

It is clear, since $\boldsymbol{\Phi}_1(z)$ is a canonical solution of ordered normal form at infinity, and since $\kappa_1 = \kappa_2 = 1$, that

$$\boldsymbol{\Phi}_0(z) \overset{\Delta}{=} \boldsymbol{\Phi}_1(z)\mathbf{K}^{-1}\sqrt{2}\begin{bmatrix} \sqrt{\frac{5}{12}} & \frac{\sqrt{6}}{6} \\ 0 & \frac{1}{2} \end{bmatrix} \tag{9.10}$$

is also a canonical solution of ordered normal form at infinity and is such that $\boldsymbol{\Phi}_0(z) = \overline{\boldsymbol{\Phi}_0(\bar{z})}$. In view of equations (9.8) and (9.10), we can therefore write equation (9.4) as

$$\boldsymbol{\Lambda}(z) = \boldsymbol{\Phi}_0(z)\tilde{\boldsymbol{\Phi}}_0(-z), \tag{9.11}$$

where

$$\boldsymbol{\Phi}_0(0)\tilde{\boldsymbol{\Phi}}_0(0) = \mathbf{I} \tag{9.12a}$$

and

$$\lim_{|z|\to\infty} z\boldsymbol{\Phi}_0(z) = \sqrt{2}\begin{bmatrix} \sqrt{\frac{5}{12}} & \frac{\sqrt{6}}{6} \\ 0 & \frac{1}{2} \end{bmatrix}. \tag{9.12b}$$

We note that Cercignani[11] has reported a factorization in the spirit of our equation (9.11). We have been unable, however, to justify some of Cercignani's results [11; pp. 84–85] since, for example, upon 'taking' determinants of his equation (311) we find an inconsistency in the number of poles on the two sides of the equality sign. We have found that the extension of scalar results to the case of matrix Riemann problems, in general, does not follow immediately[6].

10. HALF-RANGE ORTHOGONALITY AND NORMALIZATION INTEGRALS

The half-range orthogonality relations developed by Kuščer, McCormick, and Summerfield[18] for the elementary solutions of the one-speed neutron-transport equation have proved to be useful for establishing concisely the solutions to a scalar singular integral equation somewhat analogous to equation (8.1). We should thus like to prove, in a manner similar to that reported by Siewert[24] for an equation of transfer basic to the scattering of polarized light, the following theorem concerning the half-range orthogonality properties of a subset of our developed normal modes.

Theorem 5. The eigenvectors $\mathbf{F}_1(\mu)$, $\mathbf{F}_2(\mu)$, $\mathbf{F}_1(\eta, \mu)$, *and* $\mathbf{F}_2(\eta, \mu)$, $\eta \in (0, \infty)$, *are orthogonal to the related set* $\mathbf{G}_1(\mu)$, $\mathbf{G}_2(\mu)$, $\mathbf{G}_1(\eta, \mu)$ *and* $\mathbf{G}_2(\eta, \mu)$, $\eta \in (0, \infty)$, *on the half-range,* $\mu \in (0, \infty)$, *in the sense that*

$$\int_0^\infty \tilde{\mathbf{G}}(\xi', \mu)\mathbf{F}(\xi, \mu)\, e^{-\mu^2}\, \mu\, d\mu = 0, \xi \neq \xi'; \xi, \xi' = \infty \text{ or } \in (0, \infty). \tag{10.1}$$

Here $\mathbf{F}(\xi, \mu)$, $\xi = \infty$ or $\in (0, \infty)$, denotes any of the eigenvectors $\mathbf{F}_1(\mu)$, $\mathbf{F}_2(\mu)$, for $\xi = \infty$, or $\mathbf{F}_1(\eta, \mu)$, $\mathbf{F}_2(\eta, \mu)$, for $\eta \in (0, \infty)$. In a similar manner, $\mathbf{G}(\xi, \mu)$, represents either

$$\mathbf{G}_\alpha(\mu) = \mathbf{Q}(\mu)\mathbf{H}(\mu)\mathbf{H}_1^{-1}\mathbf{Q}^{-1}(\mu)\mathbf{F}_\alpha(\mu), \alpha = 1 \text{ or } 2, \tag{10.2a}$$

or

$$\mathbf{G}_\alpha(\eta, \mu) = \mathbf{Q}(\mu)\mathbf{H}(\mu)\mathbf{H}^{-1}(\eta)\mathbf{Q}^{-1}(\mu)\mathbf{F}_\alpha(\eta, \mu), \eta \in (0, \infty), \alpha = 1 \text{ or } 2. \tag{10.2b}$$

In addition

$$\tilde{\mathbf{H}}_1 = \int_0^\infty \tilde{\mathbf{H}}(\mu)\boldsymbol{\Psi}(\mu)\, \mu\, d\mu \tag{10.3}$$

and $\mathbf{H}(\mu)$ is the $\mathbf{H}$ matrix basic to our half-range analysis. In section 11 we prove the existence of a unique solution to the system of singular integral equations

$$\tilde{\mathbf{H}}(\mu)\boldsymbol{\lambda}(\mu) = \mathbf{I} + \mu P \int_0^\infty \tilde{\mathbf{H}}(\eta)\boldsymbol{\Psi}(\eta)\frac{d\eta}{\eta - \mu}, \mu \in (0, \infty), \tag{10.4a}$$

plus the constraint

$$\int_0^\infty \tilde{\mathbf{H}}(\mu)\boldsymbol{\Psi}(\mu)\,\mathrm{d}\mu = \mathbf{I}, \tag{10.4b}$$

which we take to specify $\mathbf{H}(\mu)$. As shall be shown, $\mathbf{H}(\mu)$ can be expressed in terms of $\boldsymbol{\Phi}_0(z)$, our canonical solution, of ordered normal form at infinity, of the matrix Riemann problem defined by equation (9.1); that is

$$\mathbf{H}(\mu) = \tilde{\boldsymbol{\Phi}}_0^{-1}(-\mu)\tilde{\boldsymbol{\Phi}}_0(0), \tag{10.5}$$

which can be extended to the complex plane cut along the negative real axis to yield a factorization of $\boldsymbol{\Lambda}(z)$:

$$\boldsymbol{\Lambda}(z) = \tilde{\mathbf{H}}^{-1}(z)\mathbf{H}^{-1}(-z). \tag{10.6}$$

To establish our Theorem 5, we first pre-multiply equation (4.4) by $\tilde{\mathbf{G}}(\xi', \mu)\,\mathrm{e}^{-\mu^2}/\xi$, we then post-multiply the transpose of equation (4.4), having changed ξ to ξ', by $\tilde{\mathbf{Q}}^{-1}(\mu)\tilde{\mathbf{H}}^{-1}(\xi')\tilde{\mathbf{H}}(\mu)\tilde{\mathbf{Q}}(\mu)\mathbf{F}(\xi, \mu)\exp(-\mu^2)/\xi'$, integrate the two resulting equations over μ from 0 to ∞ and subtract the two equations, one from the other, to obtain

$$\left(\frac{1}{\xi} - \frac{1}{\xi'}\right)\int_0^\infty \tilde{\mathbf{G}}(\xi', \mu)\mathbf{F}(\xi, \mu)\,\mathrm{e}^{-\mu^2}\,\mu\,\mathrm{d}\mu = \frac{1}{\sqrt{\pi}}[K_1(\xi', \xi) - K_2(\xi', \xi)],\ \xi \text{ and } \xi' > 0. \tag{10.7}$$

Here

$$K_1(\xi', \xi) = \tilde{\mathbf{M}}(\xi')\tilde{\mathbf{H}}^{-1}(\xi')\int_0^\infty \tilde{\mathbf{H}}(\mu)\tilde{\mathbf{Q}}(\mu)\mathbf{F}(\xi, \mu)\,\mathrm{e}^{-\mu^2}\,\mathrm{d}\mu \tag{10.8}$$

and

$$K_2(\xi', \xi) = \int_0^\infty \tilde{\mathbf{F}}(\xi', \mu)\tilde{\mathbf{Q}}^{-1}(\mu)\tilde{\mathbf{H}}^{-1}(\xi')\tilde{\mathbf{H}}(\mu)\tilde{\mathbf{Q}}(\mu)\mathbf{Q}(\mu)\,\mathrm{e}^{-\mu^2}\,\mathrm{d}\mu\ \mathbf{M}(\xi). \tag{10.9}$$

If now, in the manner similar to that previously reported[24], we make use of equations (4.5), (4.11), (4.13), (4.14) and (10.4) to evaluate equations (10.8) and (10.9) for all appropriate ξ and ξ', we find

$$K_1(\xi', \xi) = K_2(\xi', \xi);\ \xi' \in (0, \infty),\ \xi = \infty \text{ or } \in (0, \infty), \tag{10.10}$$

and from equation (10.7) we obtain

$$\left(\frac{1}{\xi} - \frac{1}{\xi'}\right)\int_0^\infty \tilde{\mathbf{G}}(\xi', \mu)\mathbf{F}(\xi, \mu)\,\mathrm{e}^{-\mu^2}\,\mu\,\mathrm{d}\mu = 0;\ \xi, \xi' > 0, \tag{10.11}$$

which proves the theorem. We have only established equation (10.11) formally for $\xi' \neq \infty$. However, considering that case separately, we do find that $\mathbf{G}_\alpha(\mu)$, $\alpha = 1$ or 2, is orthogonal to $\mathrm{F}(\xi, \mu)$ in the sense of Theorem 5. Of course, since $\mathbf{F}_1(\mu)$ and $\mathbf{F}_2(\mu)$ both correspond to the eigenvalue $\xi = \infty$, equation (10.1) does not ensure that the inner product, in the sense of Theorem 5, of $\mathbf{G}_1(\mu)$ with $\mathbf{F}_2(\mu)$ and $\mathbf{G}_2(\mu)$ with $\mathbf{F}_1(\mu)$ is zero.

However, for this special case we have carried out the algebra prescribed by equation (10.1) to show explicitly that

$$\int_0^\infty \tilde{\mathbf{G}}_\alpha(\mu)\mathbf{F}_\beta(\mu)\,\mathrm{e}^{-\mu^2}\,\mu\,\mathrm{d}\mu = 0,\ \alpha \neq \beta, \tag{10.12}$$

so that all of the half-range eigenvectors are orthogonal in the manner of Theorem 5.

Having established the required half-range orthogonality results, we should now like to consider again the normalized solutions given by equation (6.7) in order to present our half-range normalization integrals in a form analogous to that used for the full-range theory. We consider then the half-range adjoint set

$$\mathbf{\Theta}_\alpha(\mu) = \frac{1}{\sqrt{\pi}}\mathbf{Q}(\mu)\mathbf{H}(\mu)\mathbf{H}_1^{-1}\mathbf{Q}^{-1}(\mu)\mathbf{\Phi}_\alpha(\mu),\ \alpha = 1 \text{ and } 2, \tag{10.13a}$$

and

$$\mathbf{\Theta}_\alpha(\eta, \mu) = \mathbf{Q}(\mu)\mathbf{H}(\mu)\mathbf{H}^{-1}(\eta)\mathbf{Q}^{-1}(\mu)\mathbf{X}_\alpha(\eta, \mu),\ \eta \in (0, \infty),\ \alpha = 1 \text{ and } 2, \tag{10.13b}$$

where the vectors $\mathbf{X}_\alpha(\eta, \mu)$, $\alpha = 1$ and 2, are given by equations (6.11e) and (6.11f); we can therefore summarize our results in the manner

$$\int_0^\infty \tilde{\mathbf{\Theta}}_\alpha(\mu)\mathbf{\Phi}_\beta(\mu)\,\mathrm{e}^{-\mu^2}\,\mu\,\mathrm{d}\mu = \delta_{\alpha,\beta};\ \alpha,\beta = 1 \text{ or } 2, \tag{10.14a}$$

$$\int_0^\infty \tilde{\mathbf{\Theta}}_\alpha(\mu)\mathbf{\Phi}_\beta(\eta, \mu)\,\mathrm{e}^{-\mu^2}\,\mu\,\mathrm{d}\mu = 0,\ \eta \in (0, \infty);\ \alpha,\beta = 1 \text{ or } 2, \tag{10.14b}$$

$$\int_0^\infty \tilde{\mathbf{\Theta}}_\alpha(\eta', \mu)\mathbf{\Phi}_\beta(\eta, \mu)\,\mathrm{e}^{-\mu^2}\,\mu\,\mathrm{d}\mu = \delta_{\alpha,\beta}\delta(\eta - \eta');\ \eta,\eta' \in (0, \infty),\ \alpha,\beta = 1 \text{ or } 2, \tag{10.14c}$$

and

$$\int_0^\infty \tilde{\mathbf{\Theta}}_\alpha(\eta, \mu)\mathbf{\Phi}_\beta(\mu)\,\mathrm{e}^{-\mu^2}\,\mu\,\mathrm{d}\mu = 0;\ \eta \in (0, \infty),\ \alpha,\beta = 1 \text{ or } 2. \tag{10.14d}$$

With the half-range formalism thus established, we note that the solutions for all expansion coefficients in equations of the form

$$\mathbf{I}(\mu) = \sum_{\alpha=1}^{2} A_\alpha \mathbf{F}_\alpha(\mu) + \sum_{\alpha=1}^{2}\int_0^\infty A_\alpha(\eta)\mathbf{F}_\alpha(\eta, \mu)\,\mathrm{d}\eta,\ \mu \in (0, \infty), \tag{10.15}$$

can be expressed concisely as

$$A_\alpha = \int_0^\infty \tilde{\mathbf{\Theta}}_\alpha(\mu)\mathbf{I}(\mu)\,\mathrm{e}^{-\mu^2}\,\mu\,\mathrm{d}\mu,\ \alpha = 1 \text{ and } 2, \tag{10.16a}$$

and

$$A_\alpha(\eta) = \int_0^\infty \tilde{\mathbf{\Theta}}_\alpha(\eta, \mu)\mathbf{I}(\mu)\,\mathrm{e}^{-\mu^2}\,\mu\,\mathrm{d}\mu,\ \alpha = 1 \text{ and } 2. \tag{10.16b}$$

We note that a set of integrals of the form

$$T(\xi', \xi) = \int_0^\infty \tilde{\mathbf{\Theta}}(\xi', \mu)\mathbf{\Phi}(-\xi, \mu)\, e^{-\mu^2}\, \mu\, d\mu, \ \xi \text{ and } \xi' > 0, \tag{10.17}$$

has been evaluated and is listed elsewhere [16].

11. EXISTENCE AND UNIQUENESS OF THE H MATRIX

We first wish to prove

Theorem 6. *The equations*

$$\tilde{\mathbf{H}}(\mu)\boldsymbol{\lambda}(\mu) = \mathbf{I} + \mu P \int_0^\infty \tilde{\mathbf{H}}(\eta)\mathbf{\Psi}(\eta)\frac{d\eta}{\eta - \mu}, \ \mu \in (0, \infty), \tag{11.1a}$$

and

$$\int_0^\infty \tilde{\mathbf{H}}(\mu)\mathbf{\Psi}(\mu)\, d\mu = \mathbf{I} \tag{11.1b}$$

possess a unique solution in the class of functions continuous on every open interval of the positive real axis.

Though for the sake of brevity we do not give an explicit derivation of equations (11.1), we note that the **H** matrix specified by equations (11.1) is sufficient for establishing the half-range orthogonality theorem.

To prove Theorem VI we make use of the equivalence of the given singular-integral equations to a certain matrix version of the Riemann problem. In the manner of Muskhelishvili [21], we introduce the matrix

$$\mathbf{N}(z) = \frac{1}{2\pi i}\int_0^\infty \tilde{\mathbf{H}}(\eta)\mathbf{\Psi}(\eta)\frac{d\eta}{\eta - z} \tag{11.2}$$

which is analytic in the plane cut along the real axis and vanishes at least as fast as $1/z$ as $|z|$ tends to infinity. The Plemelj formulae [21] can be used with equation (11.2) to yield

$$\pi i[\mathbf{N}^+(\mu) + \mathbf{N}^-(\mu)] = P\int_0^\infty \tilde{\mathbf{H}}(\eta)\mathbf{\Psi}(\eta)\frac{d\eta}{\eta - \mu}, \ \mu \in (0, \infty), \tag{11.3}$$

and

$$\mathbf{N}^+(\mu) - \mathbf{N}^-(\mu) = \tilde{\mathbf{H}}(\mu)\mathbf{\Psi}(\mu), \ \mu \in (0, \infty). \tag{11.4}$$

If we make use of equations (5.10), (11.3) and (11.4) then equation (11.1a) can be cast in the equivalent form of an inhomogeneous Riemann problem:

$$\tilde{\mathbf{N}}^+(\mu) = \mathbf{G}(\mu)\tilde{\mathbf{N}}^-(\mu) + \mathbf{\Psi}(\mu)[\mathbf{\Lambda}^-(\mu)]^{-1}, \ \mu \in (0, \infty), \tag{11.5}$$

where

$$\mathbf{G}(\mu) = \mathbf{\Lambda}^+(\mu)[\mathbf{\Lambda}^-(\mu)]^{-1}. \tag{11.6}$$

The solution to equation (11.5) can be written as

$$\tilde{\mathbf{N}}(z) = \frac{1}{2\pi i}\boldsymbol{\Phi}(z)\left[\int_0^\infty \mathbf{K}(\eta)\frac{\mathrm{d}\eta}{\eta - z} + \hat{\mathbf{P}}(z)\right] \tag{11.7}$$

where $\hat{\mathbf{P}}(z)$ is a matrix of polynomials,

$$\mathbf{K}(\mu) = [\boldsymbol{\Phi}^+(\mu)]^{-1}\boldsymbol{\Psi}(\mu)[\boldsymbol{\Lambda}^-(\mu)]^{-1}, \tag{11.8}$$

and $\boldsymbol{\Phi}(z)$ is any canonical solution (of ordered normal form at infinity) to the homogeneous Riemann problem defined by equation (9.1). In order that equation (11.7) have behavior as $|z|$ approaches infinity consistent with equation (11.2), we must take $\hat{\mathbf{P}}(z)$ to be a constant matrix.

Following closely Siewert and Burniston's[25] work on the scattering of polarized light, we can now use the constraint to specify uniquely the constant $\hat{\mathbf{P}}(z)$ in equation (11.7). It thus follows that equation (11.7) along with equations (9.11) and (11.4) yields the result

$$\mathbf{H}(\mu) = \tilde{\boldsymbol{\Phi}}_0^{-1}(-\mu)\tilde{\boldsymbol{\Phi}}_0(0),\ \mu \in (0, \infty), \tag{11.9}$$

where $\boldsymbol{\Phi}_0(z)$ is the canonical solution (of ordered normal form at infinity) used in equation (9.11). We note from equation (11.9), since $\boldsymbol{\Phi}_0(z) = \overline{\boldsymbol{\Phi}_0(\bar{z})}$, that $\mathbf{H}(\mu)$ is real for $\mu \in (0, \infty)$. Further, equation (11.9) can be used to extend the definition of $\mathbf{H}(\mu)$ to the complex plane:

$$\mathbf{H}(z) = \tilde{\boldsymbol{\Phi}}_0^{-1}(-z)\tilde{\boldsymbol{\Phi}}_0(0), \tag{11.10}$$

so that the $\boldsymbol{\Lambda}$ matrix can now be factored as

$$\boldsymbol{\Lambda}(z) = \tilde{\mathbf{H}}^{-1}(-z)\mathbf{H}^{-1}(z).$$

We note that equations (5.10), (9.11) and (11.10) can be used in the Cauchy integral representation

$$\boldsymbol{\Phi}_0(z) = \frac{1}{2\pi i}\int_0^\infty [\boldsymbol{\Phi}_0^+(\mu) - \boldsymbol{\Phi}_0^-(\mu)]\frac{\mathrm{d}\mu}{\mu - z} \tag{11.11}$$

to obtain

$$\mathbf{H}(z) = \mathbf{I} + z\mathbf{H}(z)\int_0^\infty \tilde{\mathbf{H}}(\eta)\boldsymbol{\Psi}(\eta)\frac{\mathrm{d}\eta}{\eta + z} \tag{11.12}$$

or

$$\mathbf{H}(\mu) = \mathbf{I} + \mu\mathbf{H}(\mu)\int_0^\infty \tilde{\mathbf{H}}(\eta)\boldsymbol{\Psi}(\eta)\frac{\mathrm{d}\eta}{\eta + \mu},\ \mu \in [0, \infty). \tag{11.13}$$

Since we have established the existence of a unique solution to equations (11.1) and developed equation (11.13) specifically to be used, along with equation (11.1b), as the

basis for our procedure for computing the **H** matrix, it follows that we require proof of

Theorem 7. The equations

$$\mathbf{H}(\mu) = \mathbf{I} + \mu \mathbf{H}(\mu) \int_0^\infty \tilde{\mathbf{H}}(\eta)\boldsymbol{\Psi}(\eta)\frac{d\eta}{\eta + \mu}, \quad \mu \in [0, \infty), \tag{11.14a}$$

and

$$\int_0^\infty \tilde{\mathbf{H}}(\mu)\boldsymbol{\Psi}(\mu)\, d\mu = \mathbf{I} \tag{11.14b}$$

possess a unique solution in the class of functions continuous on every open interval of the positive real axis.

We have shown that equations (11.1) possess a unique solution; thus we need only show that any solution of equation (11.14a) is also a solution of equation (11.1a). We first write equation (11.14a) as

$$\mathbf{H}(\mu)\left[\mathbf{I} - \mu \int_0^\infty \tilde{\mathbf{H}}(\eta)\boldsymbol{\Psi}(\eta)\frac{d\eta}{\eta + \mu}\right] = \mathbf{I} \tag{11.15a}$$

or, alternatively,

$$\left[\mathbf{I} - \mu \int_0^\infty \tilde{\mathbf{H}}(\eta)\boldsymbol{\Psi}(\eta)\frac{d\eta}{\eta + \mu}\right]\mathbf{H}(\mu) = \mathbf{I}. \tag{11.15b}$$

If the transpose of equation (11.15b) is post-multiplied by

$$\mathbf{I} + \mu P \int_0^\infty \tilde{\mathbf{H}}(\eta)\boldsymbol{\Psi}(\eta)\frac{d\eta}{\eta - \mu}$$

then, after making use of some partial-fraction analysis and equations (4.15) and (11.15), we obtain

$$\tilde{\mathbf{H}}(\mu)\boldsymbol{\lambda}(\mu) = \mathbf{I} + \mu P \int_0^\infty \tilde{\mathbf{H}}(\eta)\boldsymbol{\Psi}(\eta)\frac{d\eta}{\eta - \mu}, \quad \mu \in (0, \infty), \tag{11.16}$$

which proves Theorem 7.

12. AN EXPEDIENT METHOD FOR COMPUTING THE H MATRIX

It is clear from the previous sections of this paper that the **H** matrix is the basic quantity required in the solutions of problems defined in terms of equation (4.18) and specified by half-range, $\mu \in (0, \infty)$, boundary conditions. It is also apparent from the analysis of section 11 that the basic proofs regarding the existence and uniqueness of the **H** matrix have been established; however, to demonstrate the utility of our analysis, we must now establish a procedure by which we can compute the **H** matrix accurately and efficiently.

As we have discussed, the **H** matrix is uniquely specified by the nonlinear equation

$$\mathbf{H}(\mu) = \mathbf{I} + \mu \mathbf{H}(\mu) \int_0^\infty \tilde{\mathbf{H}}(\eta)\boldsymbol{\Psi}(\eta)\frac{\mathrm{d}\eta}{\eta + \mu}, \ \mu \in [0, \infty), \tag{12.1a}$$

and the constraint

$$\int_0^\infty \tilde{\mathbf{H}}(\mu)\boldsymbol{\Psi}(\mu)\,\mathrm{d}\mu = \mathbf{I}. \tag{12.1b}$$

Rather than seek a numerical solution to equation (12.1a) which must also satisfy equation (12.1b), we prefer [15, 22] first to write

$$\mathbf{H}(\mu) = (1 + \mu)\mathbf{L}(\mu), \ \mu \in [0, \infty). \tag{12.2}$$

If we now substitute equation (12.2) into equation (12.1a), perform some elementary partial-fraction analysis, and make use of the constraint, equation (12.1b), then we find that $\mathbf{L}(\mu)$ must satisfy

$$\mathbf{L}(\mu) = \mathbf{I} + \mu \mathbf{L}(\mu) \int_0^\infty (1 - \eta^2)\tilde{\mathbf{L}}(\eta)\boldsymbol{\Psi}(\eta)\frac{\mathrm{d}\eta}{\eta + \mu}, \ \mu \in [0, \infty), \tag{12.3a}$$

and

$$\int_0^\infty \tilde{\mathbf{L}}(\mu)\boldsymbol{\Psi}(\mu)(1 + \mu)\,\mathrm{d}\mu = \mathbf{I}. \tag{12.3b}$$

It follows from Theorem 7 that equations (12.3) have a unique solution. We regard equations (12.3) as the basic equations to be solved numerically because an iterative procedure based on these equations has proven to converge faster than a similar iterative solution of equations (12.1).

For calculational convenience, we now prefer to make in equations (12.3) the change of variable

$$t = \frac{\mu}{1 + \mu} \tag{12.4}$$

and thus to write

$$\mathbf{L}_*(t) = \mathbf{I} + t\mathbf{L}_*(t) \int_0^1 \frac{1 - 2s}{(1 - s)^3}\tilde{\mathbf{L}}_*(s)\boldsymbol{\Psi}_*(s)\frac{\mathrm{d}s}{t(1 - s) + s(1 - t)}, \ t \in [0, 1), \tag{12.5}$$

where we have introduced the notation $g_*(t) = g[t/(1 - t)]$. We have found that equation (12.5) can be solved quite effectively by iteration.

The computations were performed in double-precision arithmetic on an IBM 370/165 computer, and we used an improved Gaussian-quadrature [17] representation of the integration process. The iterative procedure was terminated when successive calculations of $\mathbf{L}_*(t)$ differed by no more than 10^{-15}.

To substantiate confidence in our computations, several "checks" were incorporated in the calculation. As expected $\mathbf{L}_*(t)$ satisfied

$$\mathbf{I} - \int_0^1 \tilde{\mathbf{L}}_*(t)\boldsymbol{\Psi}_*(t)\frac{\mathrm{d}t}{(1 - t)^3} = \mathbf{0}, \tag{12.6}$$

an identity corresponding to the constraint, equation (12.1b), to thirteen significant figures. The equation in terms of $\mathbf{L}_*(t)$ corresponding to the identity[16]

$$2\int_0^\infty \mathbf{\Psi}(\mu)\,\mu^2 d\mu - \int_0^\infty \mathbf{\Psi}(\mu)\mathbf{H}(\mu)\,\mu\,d\mu \int_0^\infty \tilde{\mathbf{H}}(\mu)\mathbf{\Psi}(\mu)\,\mu\,d\mu = \mathbf{0} \tag{12.7}$$

was also verified to thirteen significant figures.

The analytical solution[16]

$$H(z) = \det \mathbf{H}(z) = \sqrt{\frac{12}{5}}z^2 \exp\left\{-\frac{1}{\pi}\int_0^\infty \arg \Lambda^+(\mu)\frac{d\mu}{\mu + z}\right\}, \tag{12.8}$$

where $\arg \Lambda^+(0) = -2\pi$, can be shown to satisfy

$$H(\mu) = 1 + \mu H(\mu)\int_0^\infty f(\eta)H(\eta)\frac{d\eta}{\eta + \mu},\ \mu \in [0, \infty), \tag{12.9}$$

where

$$f(\eta) = \frac{1}{3}\frac{1}{\sqrt{\pi}}e^{-\eta^2}\left[\frac{11}{2} - \eta^2 - 8\eta\, e^{-\eta^2}\int_0^\eta e^{t^2}\,dt\right]. \tag{12.10}$$

Rather than solve equation (12.9) and the appropriate constraint for $H(\mu)$ we prefer to write

$$H(\mu) = (1 + \mu)^2 L(\mu),\ \mu \in [0, \infty). \tag{12.11}$$

If we substitute equation (12.11) into equation (12.9), perform some partial-fraction analysis and make use of two identities[16] for $H(\mu)$ then we find that (after an appropriate change of variables) $L_*(t) = \det \mathbf{L}_*(t)$ must satisfy

$$\frac{1}{L_*(t)} = 1 - t\int_0^1 \frac{(1-2s)^2}{(1-s)^5} f_*(s)L_*(s)\frac{ds}{s(1-t) + t(1-s)},\ t \in [0, 1). \tag{12.12}$$

We have compared $L_*(t)$ as computed from equation (12.5) to a direct solution of equation (12.12) and the appropriate constraint to find agreement to nine significant figures.

Finally the number of quadrature points used to represent the integration process was increased to suggest that the numerical values of the $\mathbf{L}_*$ matrix given in the accompanying Table (1) were insensitive to further refinements in the quadrature scheme.

13. AN APPLICATION OF THE THEORY: THE TEMPERATURE SLIP PROBLEM

We consider the effect of a body surface on the behavior of the particle distribution function of a rarefied gaseous medium. It is known that, in the absence of boundaries, the particle distribution function in a gas with slowly varying physical parameters obeys the Chapman–Enskog equations (and therefore the macroscopic variables obey the

Table 1. The $\mathbf{L}_*$ matrix

t	$L_{*11}(t)$	$L_{*12}(t)$	$L_{*21}(t)$	$L_{*22}(t)$
0·0	1·0	0·0	0·0	1·0
0·05	1·05781	−0·0409718	−0·0406202	1·08988
0·10	1·08842	−0·0702992	−0·0691920	1·14897
0·15	1·10976	−0·0960791	−0·0939297	1·19824
0·20	1·12527	−0·119749	−0·116314	1·24158
0·25	1·13648	−0·141931	−0·136989	1·28073
0·30	1·14424	−0·162963	−0·156307	1·31664
0·35	1·14909	−0·183053	−0·174483	1·34992
0·40	1·15139	−0·202341	−0·191658	1·38097
0·45	1·15138	−0·220921	−0·207929	1·41009
0·50	1·14926	−0·238862	−0·223361	1·43748
0·55	1·14514	−0·256214	−0·238000	1·46333
0·60	1·13912	−0·273011	−0·251873	1·48774
0·65	1·13125	−0·298278	−0·264997	1·51082
0·70	1·12157	−0·305030	−0·277373	1·53265
0·75	1·11008	−0·320274	−0·288995	1·55328
0·80	1·09678	−0·335010	−0·299844	1·57276
0·85	1·08162	−0·349228	−0·309887	1·59111
0·90	1·06454	−0·362913	−0·319081	1·60835
0·95	1·04545	−0·376036	−0·327361	1·62449
0·99	1·02864	−0·386105	−0·333271	1·63660

Navier–Stokes equations). Near the body surface, the behavior of the gas is described by a rarefied Knudsen layer in which the collisional effects are only of secondary importance. It is natural to ask how the outer Chapman–Enskog (or Navier–Stokes) region can be matched consistently with the inner Knudsen layer. Saying it differently, we ask what are the velocity and temperature slip boundary conditions at a body surface for the Navier–Stokes equations due to the presence of the Knudsen layer adjacent to the body surface.

To understand the asymptotic behavior of the Knudsen layer, we may stretch locally the coordinate normal to the body surface such that the gas-kinetic motion in the Knudsen layer reduces to a locally defined half-space problem and the kinetic equation takes on a one-dimensional character in the form studied in this paper.

Since the asymptotic boundary condition of the Knudsen layer is given by the Chapman–Enskog equations, the asymptotic form of the particle distribution function is nearly Maxwellian. If we also assume that the effect of the body surface is to re-emit molecules described by a suitably chosen Maxwellian distribution and that the macroscopic variables do not vary appreciably throughout the Knudsen layer, a linearization scheme for the one-dimensional kinetic equation in the sense described in sections 2 and 3 is justified. Based on the constant collision frequency BGK model, the pertinent linearized kinetic equation for the Knudsen layer is that given by equation (3.7). The velocity-slip (or Kramers problem) for this equation for a diffusely reflecting wall has been solved exactly by Cercignani[9] and an accurate velocity-slip coefficient has been calculated,[1]. Although approximate analyses of the associated temperature slip problem have been reported by a number of authors[2, 19, 23, 28, 29], an accurate calculation of the temperature-slip coefficient has not been previously reported. Since the temperature-density effects for the problem for a diffusely reflecting wall are uncoupled from the transverse momentum effects, we will show that an accurate determi-

nation of the temperature slip coefficient for steady gas-kinetic motion may be effected by using the method of elementary solutions and the half-range expansion theorem developed in this paper for the vector integrodifferential equation (4.1).

It is straightforward to demonstrate that the heat flux in the normal direction in the Knudsen layer is a constant and since the Chapman–Enskog solution relates the heat flux linearly to the temperature gradient, to match the Chapman–Enskog region and the Knudsen layer it is only necessary to consider an asymptotic boundary condition with a constant temperature gradient. It is interesting to note that such an asymptotic boundary condition, as far as the temperature-density effects are concerned, can be satisfied by taking the asymptotic perturbation distribution function to be

$$h_{asy}(x, \mathbf{c}) = \begin{bmatrix} 1 \\ c_2^2 + c_3^2 - 1 \end{bmatrix}^T \left[\sum_{\alpha=1}^{2} A_\alpha \boldsymbol{\Phi}_\alpha(\mu) + \sum_{\beta=3}^{4} A_\beta \boldsymbol{\Psi}_\beta(x, \mu) \right], \tag{13.1}$$

where the $\boldsymbol{\Phi}_\alpha'$s and $\boldsymbol{\Psi}_\beta'$s are the discrete solutions to equation (4.1), and c_2, c_3 are the components of the dimensionless particle velocity in the transverse directions. Since the Chapman–Enskog theory requires the medium to obey the perfect gas law and the pressure in the Knudsen layer far from the wall is a specified constant, we deduce from the definitions

$$n_{asy}(x) \overset{\Delta}{=} \pi^{-3/2} \int h_{asy}(x, \mathbf{c})\, \mathrm{e}^{-c^2}\, \mathrm{d}^3 c, \tag{13.2}$$

and

$$T_{asy}(x) \overset{\Delta}{=} \tfrac{2}{3}\pi^{-3/2} \int h_{asy}(x, \mathbf{c})(c^2 - \tfrac{3}{2})\, \mathrm{e}^{-c^2}\, \mathrm{d}^3 c, \tag{13.3}$$

the requirements

$$A_2 = -\sqrt{\tfrac{2}{3}} A_1 \text{ and } A_4 = -\sqrt{\tfrac{2}{3}} A_3. \tag{13.4}$$

For an asymptotic temperature gradient of unity, it is clear that the temperature slip coefficient ξ defined by $T_{asy}(0) = \xi(\mathrm{d}/\mathrm{d}x) T_{asy}(x)|_{x=0}$ will be given by $\xi = \epsilon' l_t$, [19], where

$$\epsilon' = \sqrt{\tfrac{2}{3}} A_1, \tag{13.5}$$

$l_t = (4/5)(\mathcal{K}/nk)(m/2kT)^{1/2}$ is a mean free path, and $\mathcal{K}$ is the thermal conductivity.

For a diffusely reflecting wall, the boundary condition at $x = 0$ requires

$$\begin{bmatrix} B \\ 0 \end{bmatrix} = \sum_{\alpha=1}^{4} A_\alpha \boldsymbol{\Phi}_\alpha(\mu) + \sum_{\alpha=1}^{2} \int_0^\infty A_\alpha(\eta) \boldsymbol{\Phi}_\alpha(\eta, \mu)\, \mathrm{d}\eta, \ \mu \in (0, \infty). \tag{13.6}$$

The unknown constant B is related to the density of the gas near the wall and need not be specified for temperature-slip coefficient calculations. We make use of equation (13.4) and the specified asymptotic temperature gradient to write equation (13.6) as

$$\sqrt{\tfrac{3}{2}} \boldsymbol{\Phi}_3(\mu) - \boldsymbol{\Phi}_4(\mu) = A_1 \boldsymbol{\Phi}_1(\mu) + (A_2 - B) \boldsymbol{\Phi}_2(\mu) + \sum_{\alpha=1}^{2} \int_0^\infty A_\alpha(\eta) \boldsymbol{\Phi}_\alpha(\eta, \mu)\, \mathrm{d}\eta. \tag{13.7}$$

Theorem 3 and equation (10.16a) enable us to solve equation (13.7) for A_1:

$$A_1 = \int_0^\infty \tilde{\Theta}_1(\mu)[\sqrt{\tfrac{3}{2}}\Phi_3(\mu) - \Phi_4(\mu)]\, e^{-\mu^2}\, \mu\, d\mu. \tag{13.8}$$

It is clear from the definition of $\Theta_1(\mu)$ that A_1 may be expressed in terms of appropriate moments of the **H** matrix discussed in section 11. The numerical procedure used to evaluate integrals involving the **H** matrix was given in section 12. We find

$$\epsilon' = \tfrac{5}{8}\pi^{1/2}(1{\cdot}17597). \tag{13.9}$$

This compares with the variational result of $\epsilon' = \frac{5}{8}\pi^{1/2}$ (1·1621), [2, 19, 23], Wang Chang and Uhlenbeck's result of $\epsilon' = \frac{5}{8}\pi^{1/2}$ (1·150), [28], and Welander's value of $\epsilon' = \frac{5}{8}\pi^{1/2}$ (1·173), [29]. We believe our result to be accurate to the number of significant figures quoted.

Acknowledgement—The authors are grateful to Professor E. E. BURNISTON for many valuable discussions during the course of this investigation. This research was supported in part by NSF grants GK-11935 and GU-1590, and NASA grant NGL34–002–084, and the Triangle Universities Consortium on Air Pollution (comprising Duke University, North Carolina State University and the University of North Carolina at Chapel Hill), which is funded by the Office of Manpower Development of the U.S. Environmental Protection Agency.

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(*Received* 8 *June* 1972)

Résumé—La méthode des solutions élémentaires est employée pour résoudre deux équations integro-différentielles couplées pour la détermination des effets du rapport température-densité, dans un modèle BGK linéarisé dans la théorie cinétique des gaz. L'état complet de tout le domaine et les théorèmes d'orthogonalité sont démontrés pour les modes normaux développés et la fonction de Green pour un milieu infini est établie comme illustration du formalisme sur tout le domaine.

Le problème de Riemen approprié à une matrice homogène est étudiée et l'état complet de tout le domaine et les théorèmes d'orthogonalité sont démontrés pour un certain sous-système des modes normaux. Les théorèmes nécessaires d'existence et d'unicité concernant la matrice H, fondamentale pour l'analyse du domaine complet, sont démontrés et une méthode de calcul précise et efficace est présentée. Le problème du glissement de température d'un demiespace est résolu analytiquement et une valeur très précise du coefficient de glissement de température est donnée.

Zusammenfassung—Das Verfahren elementarer Lösungen wird verwendet, um zwei gekuppelte Integro-differentialgleichungen zu lösen, die genügen, um die Temperatur-Dichtewirkungen in einem linearisierten BGK-Modell in der kinetischen Theorie von Gasen zu bestimmen.

Vollbereichs-Vollständigkeits- und Orthogonalitätstheoreme werden für die entwickelten Normalformen bewiesen und Green's Funktion des unendlichen Stoffes wird als eine Illustration des Vollbereichsformalismus konstruiert.

Das entsprechende Riemann'sche Problem einer homogenen Matrix wird besprochen und Halbbereichs-Vollständigkeits- und Orthogonalitätstheoreme werden für eine bestimmte Untergruppe der Normalformen bewiesen. Die für die H-Matrix belangreichen erforderlichen Existenz- und Eindeutigkeitstheoreme, grundlegend für die Halbbereichs-Analyse, werden bewiesen, und eine genaue und wirksame Berechnungsmethode wird besprochen. Das Temperaturgleitproblem des Halbraumes wird analytisch gelöst und ein sehr genauer Wert des Temperaturgleitkoeffizienten wird berichtet.

Sommario—Il metodo delle soluzioni elementari viene usato per risolvere due equazioni integrodifferenziali accoppiate, sufficienti a determinare gli effetti della densità di temperatura in un modello BGK linearizzato nella teoria cinetica dei gas.

Vengono dimostrati i teoremi della completezza e dell'ortogonalità su tutta la gamma per i modi normali sviluppati e, per illustrare il formalismo sull'intera gamma, viene costruita la funzione di Green per un mezzo infinito.

Viene discusso il problema di Riemann per una matrice omogenea appropriata e, per un certo sottoassieme dei modi normali, vengono dimostrati i problemi di completezza e ortogonalità per la semigamma. Vengono dimostrati i necessari teoremi di esistenza e unicità relativi alla matrice H, fondamentali per l'analisi della semigamma, e viene discusso un metodo di calcolo accurato ed efficiente. Il problema temperatura/scorrimento del semispazio viene risolto analiticamente e viene dato un valore molto accurato del coefficiente temperatura/scorrimento.

Абстракт — Применен метод элементарных решений для решения двух интегро-дифференциальных уравнений достаточных для определения эффектов температуры и плотности в линеаризованной модели (BGK) в кинетической теории газов. Даны доказательства теорем об общей полноте и ортогональности для развиваемых нормальных режимов, построена функция Грина для бесконечной среды в качестве иллюстрации общего формализма. Обсуждена соответствующая римановская проблема однородной матрицы, даны доказательства теорем о половинной полноте и ортогональности для определенной подсистемы нормальных режимов. Доказаны искомые теоремы о существовании и единственности относительно матрицы *H*, лежащие в основе анализа половинной полноты, а также обсужден точный рациональный метод вычисления. Аналитически решена проблема полупространственного скольжения и температуры, сообщено высокоточное значение коэффициента скольжения и температуры.

PHYSICAL REVIEW A VOLUME 9, NUMBER 6 JUNE 1974

Tricritical points in multicomponent fluid mixtures*

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(Received 2 July 1973)

In view of experimental considerations, we give a model-independent argument that the novel tricritical points in multicomponent fluid mixtures, where three phases simultaneously become critical, are points on the boundary of a *single two-dimensional surface of critical points*. This result is corroborated by the Landau model suggested by Griffiths. The relationship between these tricritical points and the complex "higher-order" critical points proposed to exist in certain magnetic systems is elucidated.

I. INTRODUCTION

In 1970 Griffiths[1] proposed the concept of a tricritical point as being the point of intersection of three lines of critical points in a phase diagram using intensive thermodynamic variables. He further suggested, as examples, metamagnets,[2] He^3-He^4 mixtures,[3] and ammonium chloride.[4] There has also been considerable speculation[1,5] that similar points might exist and might be found in the phase diagrams of complex fluid mixtures. That such points have already been proposed and, indeed, that they had been investigated prior to 1970 has recently been pointed out by Widom and Griffiths.[6-8] A full and very complete discussion of experimental evidence for these points has been given in Refs. 6–8, and we refer the reader to these papers for details.

A different way in which critical points more complex than tricritical points can occur has been shown to involve intersecting lines of tricritical points, and a classification of critical points has been introduced to differentiate these points from tricritical points or ordinary critical points.[5] The question has been raised[6,8] as to how the new points discovered in fluids are related to such a classification.

Properties of the phase diagram[9] were the basis of the approach we suggested previously.[5,10] In particular, we emphasized the importance of the connectivity properties of different spaces of critical points and tricritical points. Accordingly it is the connectivity of the different critical points of multicomponent fluid mixtures in the space of truly intensive or "field" variables that we emphasize here.

II. DEMONSTRATION THAT ALL CRITICAL POINTS FORM A SINGLE CONNECTED SURFACE

The basic idea is to consider a system where three distinct phases can be in equilibrium. These might be three liquids or two liquids and a vapor phase. On changing the thermodynamic variables (temperature, pressure, chemical potentials of different components) one pair of phases will become critical, in the presence of the third. In a binary system the point where this occurs is the end point of a line of critical points which bounds the surface of points where the two phases coexist. There are no degrees of freedom and such a point is unique in the phase diagram.[7]

In a ternary, quaternary, or more complicated system this point has one or more degrees of freedom. Thus a line of "critical end points" is possible. Such a line is the boundary of a surface of critical points where two phases are critical.

In the particular systems of interest it is possible, by varying physical conditions, to make a different pair of the three phases become critical in the presence of the third, thus producing a second line of critical end points.

Finally, by achieving exactly the correct physical conditions it is possible for all three phases to become critical simultaneously. In a ternary system such a point is unique; there are zero degrees of freedom. Here, only this and similarly simple cases are considered.

Experimentally there is the following arrangement. A tube containing a three-component mixture with a three-phase system is cooled to observe the appearance of successive phases. We

will call these phases α, β, and γ, where for purposes of argument $\rho_\alpha < \rho_\beta < \rho_\gamma$. The ratio of the various components is varied until the second meniscus appears via a critical mode. This could be the lower meniscus in which case phases γ and β become critical at a lower temperature. If pressure, temperature, and one other intensive parameter are allowed to vary, a phase diagram of the type shown in Fig. 1(a) will be observed. At the point P, the phases γ and β become critical in the presence of a third phase α. If pressure is increased, the lightest phase α will disappear and a line of critical points between phases γ and β will develop. For increasing temperatures above P, there will be a line of critical points bounding the surface of coexistence points which separates the region of light phase α from the region of the heavier phase $\gamma\beta$.

By varying the ratios of components appropriately, it is possible, in the physical systems of interest, to make the upper meniscus, separating phase β from phase α, appear second on cooling. The corresponding phase diagram is shown in Fig. 1(c). Again there is a special point P' which is the end point of a line of critical points for phases β and α, and which is also the end point of the line of points where three phases coexist. If the temperature is increased above P' then the coexistence surface separating phase γ from the combined phases $\beta\alpha$ will terminate in a line of critical points.

If the transition to the phase diagram of Fig. 1(c) happens by a continuous variation from Fig. 1(a) then there must be a situation where both menisci become critical simultaneously.

The corresponding phase diagram is shown in Fig. 1(b). It may be seen that the point P of Fig. 1(a) has migrated along the coexistence surface to the point P_t on the boundary. At the point P_t all three lines of critical points, a ($\gamma\beta$ critical), b ($\gamma\alpha$ critical), and c ($\beta\alpha$ critical) meet.

The purpose of constructing Figs. 1(a)–1(c) is to consider the behavior of the critical surface in the immediate neighborhood of the tricritical point P_t, and the figures should therefore be understood to represent the phase diagram close to P_t and not far away from the tricritical point. Also, since our primary interest is the connectivity of the surface of critical points, and not its specific shape, we do not have to choose the variables precisely, since the connectedness would remain unchanged even if different variables were chosen. [Similar remarks apply to Figs. 2(a) and 2(b) in what follows.]

We can now demonstrate that the lines of critical points a, b, c in Figs. 1(a)–1(c) form a single continuous surface of critical points bounded by the line of critical end points P-P_t-P'. Consider a

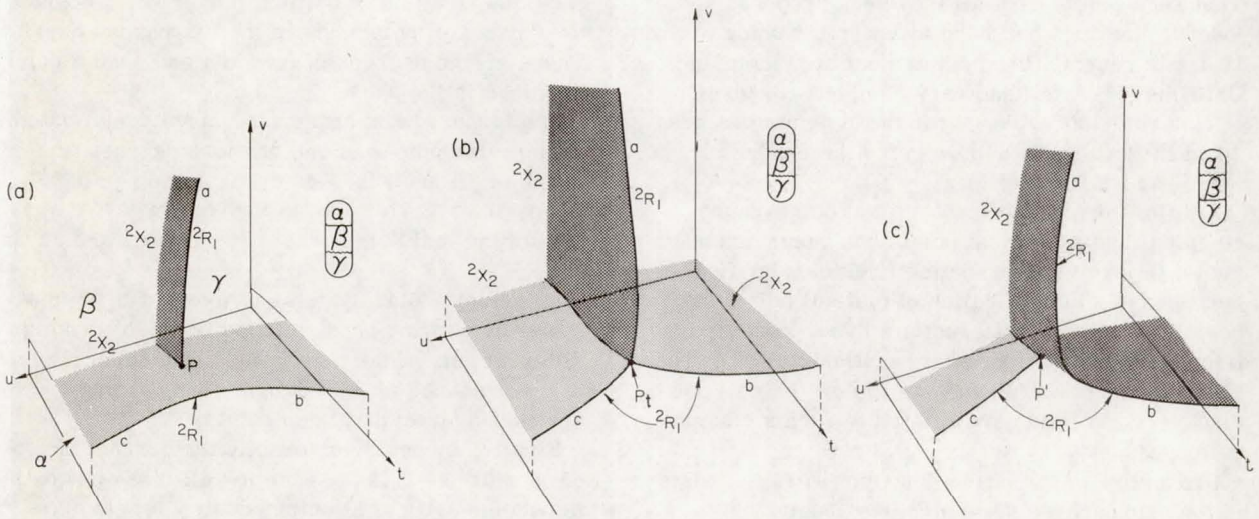

FIG. 1. Three-dimensional subspaces of the full four-dimensional space of field variables of a three-component system (or of five dimensions for four components). The variable t may be thought of as the temperature, the variable v as the pressure, and the third variable u as a suitable combination of chemical potentials. We use the notation of Ref. 5 to indicate lines and surfaces of critical points in the diagram. Lines of critical points are indicated $^2R_1$ (order 2, dimension 1). Coexistence surfaces are indicated by $^2X_2$ (two phases and dimension 2). (a) Section containing a point P where the two heavier phases γ, β are critical in the presence of the lightest α, as represented by the schematic tube showing the $\gamma\beta$ meniscus critical and the $\beta\alpha$ meniscus stable. (b) Section through the point P_t where all three phases become critical simultaneously, the $\gamma\beta$ and $\beta\alpha$ menisci are simultaneously critical in the schematic tube. (c) Section containing a point P' where the lighter phases β, α are critical in the presence of the heaviest γ, as shown in the schematic representation.

point on line a in Fig. 1(a); move continuously through Figs. 1(b) to 1(c). Now move along the line a-b to a point at the "b" end, and now move along continuously through Figs. 1(b) to 1(a). Still being on line b, we can move along the line b-c to the "c" end. Now move continuously through Figs. 1(b) to 1(c) and our assertion is demonstrated. We have not passed through the point P_t and so the lines a, b, c form sections of a single continuous surface of critical points bounded by lines of critical end points $P - P_t - P'$.

The fact that there is only a single surface of critical points is also corroborated by the model of Griffiths[8] (see his Fig. 3). This fact is in strong contrast to the metamagnet where it is not possible to go from a point on the wing boundaries to a point on the physical critical line without passing through a tricritical point[1,10] and the spaces of critical points are distinct and separate.

It will be plausible to conjecture that since all the critical points form a single surface, the critical-point exponents are the same at all critical points except the tricritical point itself. In contrast, for a metamagnet no reasons have been presented for why the critical-point exponents should be the same along all three of the critical lines meeting at the tricritical point. However, it has been suggested[11] that for most tricritical points the existence of a hidden variable linking the wing boundaries to the physical critical line (the variable called a_3 by Griffiths[8]) might force the equality of exponents.

III. MODEL SURFACE OF CRITICAL POINTS

The phase diagrams of Figs. 1(a)–1(c) were three dimensional and so parametrizing them with an extra field variable introduces a fourth dimension to the phase diagram. The connectivity, and other properties, of the surface of critical points formed by the critical lines a, b, c of Fig. 1, are most easily studied in a three-dimensional subspace of the full phase diagram which contains the whole critical surface. The shape of the critical surface in such a subspace may be determined as follows.

Consider the critical lines as they appear on the paper in Figs. 1(a)–1(c) and consider how these lines would form a smooth surface in three dimensions if the extra parameter were used to plot the height of the paper, with Fig. 1(c) above 1(b) above 1(a). By this combined projection and motion we generate a single connected surface of critical points with a boundary formed by the line of points P-P_t-P'.

A surface, which is topologically equivalent to the surface of critical points thus obtained, is shown in Figs. 2(a) and 2(b). Fig. 2(a) is a contour map of the surface, and the heights h of the contours are given by the hyperbolae $xy = h$. The boundary of the surface P-P_t-P' is represented by the parabola $y = -cx^2$ in the lower half of Fig. 2(a). The topological equivalence of the surface of Fig. 2(a) to the surface of critical points may be seen as follows.

Consider a section of Fig. 2(a) at constant height h. If $h > 0$ there are two hyperbolae, one in the upper right quadrant and one which terminates on the portion of the parabola labeled P in the lower left. This is a representation of Fig. 1(a) for which there are two critical lines, one labeled (a) terminating at P and another labeled (b-c).

If $h < 0$ there are again two hyperbolae, one in the upper left quadrant which corresponds to the line of critical points (a, b) of Fig. 1(c), and one

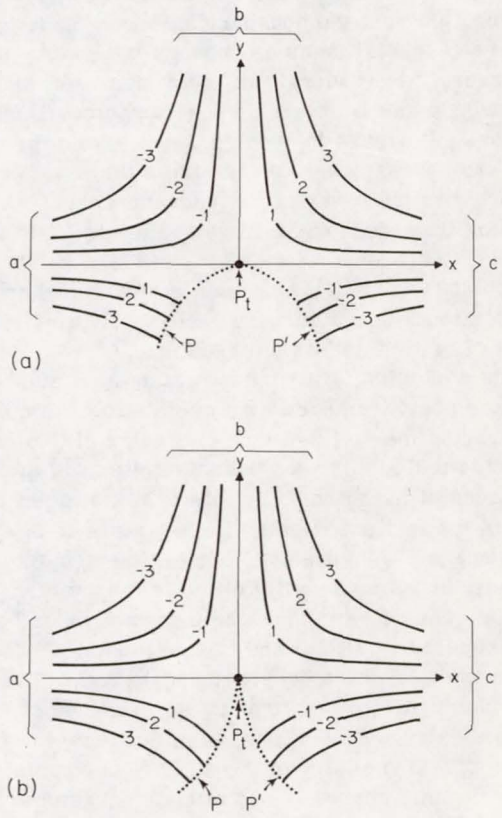

FIG. 2. Section of the four-dimensional space of field variables containing the full surface of critical points. The surface is represented by a contour map of hyperbolae and is fully explained in the text. The variables x and y are suitable field variables. (a) Boundary of the surface is smooth at P_t. The lines of critical end points P, P' form a smooth line in the four-dimensional space. (b) Boundary of the surface is cusplike at P_t. The lines of critical end points P, P' form a cusp in the four-dimensional space.

in the lower right quadrant of Fig. 2 which corresponds to the critical line c terminating at P in Fig. 1(c).

When $h=0$ the hyperbolae degenerate into the three axes, for $x<0$, $y>0$, and $x>0$ corresponding to the lines a, b, c of Fig. 1(b) which terminate at P_t.

Accordingly, in Fig. 2, the ends of the hyperbolae are labeled a, b, c according to the parts of the critical lines in Fig. 1 to which they correspond.

The complete topological correspondence between the sections of the surface in Fig. 2 and the critical lines of Fig. 1 is therefore clear. The points P, P_t, P' are a line forming a boundary of the surface of critical points, and the point P_t, which is the tricritical point, corresponds to a saddle point of the surface in the projective space of Fig. 2.

It may seem that a very special space has been chosen, and that the boundary has been made to go in a very special fashion—through the saddle point. However, this is merely in accordance with the following general physical requirements: (i) Only one point P occurs in each phase diagram; therefore the boundary *has* to pass from the lower left quadrant to the lower right quadrant without passing through the upper two quadrants; (ii) Critical lines only split or end at a point like P. Thus the point P *has* to pass through the origin where the section would otherwise necessarily give four lines of critical points intersecting.

This analysis of the critical points as a single surface provides another viewpoint from which to understand the fact that only two pairs of the three possible pairs of phases became critical in the presence of the third. The simplest viewpoint is that of the test tube itself. If the phases α, β, γ are ordered in increasing density, then (α, β) can be critical (same density) in the presence of γ, and $(\beta\gamma)$ can be critical in the presence of α, but $(\alpha\gamma)$ cannot be critical and of equal density without the phase β having a density equal to both. Thus the possibility $(\alpha\gamma)$ critical in the presence of β is eliminated. Widom[7] has related this fact to the geometric asymmetry of the solid figure containing three distinct phases α, β, γ at constant temperature less than the tricritical temperature.

In terms of the phase diagrams in spaces of truly intensive variables (Widom used densities, or extensive variables), the existence of two lines of critical end points instead of three has a very simple topological interpretation: the boundary of a single surface is locally divided into *two* separate parts by the removal of a single point. Thus the tricritical point P_t divides the boundary of the surface of critical points into two lines of critical end points but *cannot* divide it into three different lines of critical end points.

IV. SPECIAL DIRECTIONS

It was shown by Griffiths[8] that in his model there are four variables of scaling, each with different exponents at the special point. From a purely phenomenological point of view one can define four different directions at the special point P_t in the same spirit as Griffiths and Wheeler.[9] From Fig. 2 it may be seen that these directions are (i) the limiting "strong" direction for the surface of critical points, (ii) the limiting "weak" direction for the surface of critical points, (iii) the tangent to the line of critical end points at the tricritical point, and (iv) the limiting second direction parametrizing the surface of critical points. These correspond to the variables called a_1, a_2, a_4, and a_3, respectively, by Griffiths.[8]

V. TRANSLATION TO COMPOSITION VARIABLES

The variables over which the experimentalist has easy control are unfortunately not the intensive field variables like the chemical potentials, but only the densities. In these variables the phase diagrams have been discussed by Widom.[7] It has been pointed out by Griffiths[8] that the precise composition of the tricritical point P_t probably does not coincide with the composition of the regions of coexistence of three phases at temperatures smaller than the temperature at P_t.

Such behavior has probably been observed in the system carbon-dioxide-methanol-water because when a constant volume specimen (i.e., a sealed tube) of precisely the correct composition is increased in temperature, one does not observe the simultaneous disappearance of two menisci. Instead[12] one meniscus disappears critically and simultaneously a second appears critically. This remarkable behavior does not change any of our geometric conclusions, because it can be interpreted as follows: the constant volume, constant composition path does not follow the line of points where three phases coexist in Fig. 1(b), but rather it passes from one coexistence surface (γ, β) to another (α, β) directly through the point P_t.

VI. RELATIONSHIP TO OTHER COMPLEX SYSTEMS

In previous work[5,10] we have given several examples of complex magnetic systems and we have attempted to systematically classify all the coexistence and critical points in such systems. For

points where several phases coexist without being critical this is relatively simple, since the appropriate quantities are the number of coexisting phases and the dimensionality of the space. These are related to the total number of thermodynamic intensive variables by the phase rule.[5]

For critical points, it can be argued that every phase that becomes critical after the first two implies the loss of an extra degree of freedom in addition to the one lost because of coexistence.[7] Hence if there is a system with n thermodynamic variables possessing a point where p phases are in coexistence, of which q are critical (altogether rather than in two separate groups q_1 and q_2, although the generalization to such cases is simple). The dimensionality of the space on which this occurs will be given by

$$d = n - (p-1) - m, \tag{1}$$

where $m = \max(q-1, 0)$ since the case $q=1$ is not meaningful and $m=0$ if $q=0$. As a special case if all p of the phases are critical we obtain $d = n+2-2p$.

While this equation holds for fluids, it is violated by the original tricritical points, for which $n=p=3$, $d=0$ and also by certain magnetic models[10] which contain highly symmetric points where lines of tricritical points intersect, and $n=p=4$ and $d=0$. Before giving the equation which correctly describes both fluids and the complex magnetic systems let us contrast the two ways of classifying more complicated critical points that have been proposed.

For fluids Widom[7] has proposed that the important quantity is the number of phases becoming critical, and that this number should be used as the order of the critical points.

For complex magnetic systems we have proposed an apparently different scheme which is based on the original proposal for tricritical points[1] where three different lines of critical points intersected. Accordingly we gave examples of systems where different lines of tricritical points intersected, and gave the points of intersection an order different from (one larger than) that of tricritical points. Because of the symmetry of the various systems we investigated, there were no lines of critical end points, i.e., points where one or more phases coexist with others that are critical. Consequently the number of variables n needed to obtain a point where four phases are simultaneously critical was reduced from six to four.

For tricritical points in fluids, it is possible to artificially reduce the number of variables and eliminate the lines of critical end points from the phase diagram. For example, Fig. 1(b) is an illustration of this since it is a three-dimensional section of the four-dimensional field space with all lines of critical points ending at the tricritical point. Similarly the $h=0$ section of Fig. 2 produces the same result. Griffiths[8] has shown how a similar phase diagram can be obtained by taking the section $a_3=0$ of his four-dimensional phase diagram with variables a_1, a_2, a_3, a_4.

Let us now return to the case of the intersecting lines of tricritical points in the variable interaction metamagnet. It has been shown[10] that the point of intersection (the point of order 4) is a point where four phases become simultaneously critical. Thus in this case the definition of order suggested by us coincides with the definition suggested by Widom.[7] This fact may be generalized, because the only reason there should be more than one line of tricritical points is because there are more than three phases available. The different lines of tricritical points will intersect at points where more than three phases become simultaneously critical.

To obtain a version of Eq. (1) which is satisfied by all the cases considered so far, it is necessary to consider the number of variables which possess nonzero scaling power at the point under consideration. It is important to note that this number may be less than the number of significant directions picked out by the phase diagram. For example, on an ordinary line of critical points three directions are determined, but only two (the strong and weak) are associated with variables which scale. Alternatively, on the line of critical end points P, P_t, P', all four of the directions (i)–(iv) are defined but only (i) and (ii) are associated with variables which scale (except at P_t where all four enter the scaling equation). The implications of the simple Landau model[8] are that for fluids the number of scaling directions $s = 2(q-1)$. Another quantity that is important is the number of phases which are in equilibrium but which are not critical, $x = p - q$. In terms of the variables s and x, Eq. (1) may be rewritten as[13]

$$s + x + d = n. \tag{2}$$

It will be seen that this equation also holds for the old tricritical points, and the intersection of lines of tricritical points. It is satisfied by construction from (1) by all the points in generalizations of Widom's scheme for which the order is given by $\mathcal{O} \equiv q = \frac{1}{2}s + 1$ and by all the points in our scheme[5] for which the order is given by $\mathcal{O} = s$. These two possibilities express, respectively, the maximum and minimum number of scaling variables at a point where $\mathcal{O}$ phases became critical, $\mathcal{O} \leq s \leq 2(\mathcal{O}-1)$. For $\mathcal{O}=2$ there is only one possibility, $s=2$; for $\mathcal{O}=3$ there are two cases, $s=3, 4$; and for $\mathcal{O}>3$ there are many possibilities.

Thus we have proposed[5] a minimal scheme for complex critical points whereas Widom and Griffiths have proposed a maximal scheme.

In conclusion it should be reiterated that while the definition of order for critical points suggested in Ref. 5 does not appear to be applicable to fluids, it is consistent with the definition in terms of the number of phases becoming critical.[7]

ACKNOWLEDGMENT

We would like to thank Professor B. Widom and Professor R. B. Griffiths for drawing our attention to the references cited in the Soviet literature in this manuscript and for stimulating conversations. Also thanks are due to R. B. G. for comments on a previous version of this paper.

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*Supported in part by the U. S. Atomic Energy Commission, the National Science Foundation, the National Aeronautics and Space Administration, the Office of Naval Research, and the Air Force Office of Scientific Research.

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PHYSICAL REVIEW B VOLUME 8, NUMBER 1 1 JULY 1973

Generalized Scaling Hypothesis in Multicomponent Systems. I. Classification of Critical Points by Order and Scaling at Tricritical Points*

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The goal of this work is to provide an analysis of spaces of critical points for multicomponent systems. First, we propose the geometric concept of order $\mathcal{O}$ for critical points; we distinguish it from a previous definition of a "multicritical" point. Specifically, we may define the intersection of spaces of critical points of order $\mathcal{O}$ to be a space of critical points of order $(\mathcal{O}+1)$. Ordinary critical points are defined to be of order $\mathcal{O}=2$, so that the tricritical points introduced by Griffiths are of order $\mathcal{O}=3$. We discuss more general examples of critical spaces of order $\mathcal{O}=3$ which are known for a wide variety of systems; we also propose several examples of models of magnetic systems showing critical points of order $\mathcal{O}=4$—i.e., systems having *intersecting lines of tricritical points*. The analysis of critical and coexistence spaces also provides a new form of the Gibbs phase rule suitable for complex magnetic models. Next we define—for the critical points of order $\mathcal{O}$ of which examples have been given—special directions in terms of which to make a scaling hypothesis. We give the hypothesis for simple systems and then for tricritical points, and then, in a subsequent paper, part II, the special directions are used to make a scaling hypothesis at spaces of critical points of any order. Certain predictions (e.g., scaling laws and "single-power" scaling functions) follow in a simple and straightforward fashion. We consider the scaling hypothesis at a critical space of order $\mathcal{O}$ in terms of a group of transformations. We can define a set of invariants of the group. It is possible, for $\mathcal{O}\geq 3$, to make a *second* scaling hypothesis for the space of order $\mathcal{O}-1$ using certain of these invariants as *independent* variables. This is advantageous because certain "double-power" scaling functions then follow directly; these predict that for $\mathcal{O}=3$, experimental data collapse from a *volume* onto a *line*. This prediction is to be contrasted with ordinary scaling functions, which predict that data collapse by only a single dimension (e.g., from a volume onto a surface or from a surface onto a line).

I. INTRODUCTION: THE ORDER OF A CRITICAL POINT

The purpose of this work is (i) to propose the concept of the "order" of a critical point, (ii) to give examples of critical points of orders three and four, and (iii) to present a form of the scaling hypothesis for spaces of *arbitrary* order. The work is divided into two parts. In this paper (I), we focus upon concrete examples illustrating critical points and scaling at critical points of order three, while in a subsequent paper[1] (II), we consider scaling for spaces of arbitrary order. Firstly, we must develop the concept of the order of a critical point, and that is the task of this section.

The scaling hypothesis was originally formulated for the critical point of a simple magnet and a simple fluid.[2-4] These systems each have two purely intensive variables [(H, T) and (P, T), respectively]. Such variables we call *fields*, adopting the terminology of Griffiths and Wheeler.[5]

A very wide variety of physical systems whose critical phenomena are under active study have more than two field variables; two common examples are antiferromagnets and binary mixtures. In such systems, one can have lines (or, in general, spaces of dimension larger than one) of critical points. Recently, special attention has come to focus on those systems for which *three* lines of critical points intersect, and the point of intersection has been called a *tricritical point* by Griffiths.[6]

The scaling hypothesis has recently been extended[7] to treat some (but not all) aspects of this novel type of "critical point". In this work we present a comprehensive scaling treatment of general multicomponent systems. First we give a detailed treatment of tricritical points. Our approach is then generalized to more complex situations.

One example of a more complex situation is a system for which *four* lines of critical points intersect; in a natural extension of Griffiths's terminology, Nagle and Bonner[8] have called such points tetracritical points. We show here that, in the particular case studied by those authors, *the tetracritical point is qualitatively the same as a tricritical point* in the sense that the formulation of the scaling hypothesis there is the same as at tricritical points.

Qualitatively different points ("spaces")[9] can be achieved in systems with more than three field variables; a more general scaling hypothesis is needed and correspondingly more predictions are obtained. These are discussed in detail in Paper II.

Two simple examples of systems with more than three field variables are provided by He^3-He^4 and ammonium chloride, and both these systems

show *lines* of tricritical points.

Liquid He^3-He^4 mixtures have thermodynamic variables $[P, T, \mu_4 - \mu_3, \eta]$, where P denotes pressure, T denotes temperature, μ_4 and μ_3 are the chemical potentials of He^4 and He^3, respectively, and η is a variable conjugate to the superfluid density. The "λ line" in the P-T plane becomes a two-dimensional surface of singularities with increasing mole fraction of He^3. This surface terminates at a line of special points,[10(a)] which is in fact a line of tricritical points.

Ammonium chloride possesses an order-disorder transition for which the transition temperature increases with increasing pressure, and changes from first order to second order at a tricritical point. If one replaces some of the hydrogen by deuterium in the ammonium group,[10(b)] then the position of the tricritical point (and the whole line of order-disorder transitions) changes. The variables are thus $(P, T, \mu_H - \mu_D, \eta)$, where η is a variable conjugate to the order parameter.

In a system of sufficient complexity, several lines of tricritical points can occur. A point of intersection of lines of tricritical points is qualitatively different from a point where lines of ordinary critical points intersect. This should be clear from the topology of the situation: At a line of tricritical points, surfaces of critical points meet, while at a point where lines of tricritical points intersect, several surfaces of critical points (bounded on each side by the tricritical lines) converge on the point.

To distinguish such points—and in general spaces of such points—we will refer to them as critical points of higher order, and we will associate a number with each order as follows. We define ordinary critical points to be of order $\mathfrak{o} = 2$; then a critical point (or space of points) of order $\mathfrak{o}+1$ ($\mathfrak{o} \geq 2$) is defined to be a special point where lines (or spaces) of points of order $\mathfrak{o}$ intersect. Thus a tricritical point is of order $\mathfrak{o}=3$ and a point of intersection of lines of tricritical points is of order $\mathfrak{o}=4$.

Griffiths and Wheeler[5] reasoned that the dimensionality of a space of ordinary critical points (of order $\mathfrak{o}=2$) is $(n-2)$. In the systems we consider below the dimensionality, d, of a space of critical points of order $\mathfrak{o}$ is always one less than the dimensionality of the spaces of critical points of order $(\mathfrak{o}-1)$ which intersect at it. Therefore, we find the value of d for arbitrary $\mathfrak{o}$ by induction from $\mathfrak{o}=2$ to be, in these cases,

$$d = n - \mathfrak{o}, \tag{1.1}$$

where n is the total number of field (and fieldlike)[11] variables available.

Critical points of complex thermodynamic systems can also be analyzed by making the scaling hypothesis from the outset. The significant quantity is then the number of relevant scaling variables. Using the renormalization group, it has been suggested[12] how more than two relevant scaling variables can occur, but the geometry of the phase diagram was not considered at all. In most of the examples known to the authors, the number of significant scaling directions is equal to the order. This may not always be true for more complex systems [e.g., fluid mixtures, for which Eq. (1.1) may need modification, becoming $d \leq n - \mathfrak{o}$].

A specific example which demonstrates the importance of distinguishing the order of a critical point from the number of critical lines meeting there is the tetracritical point. This is a point of order $\mathfrak{o}=3$ with true field variables (H, H_2, T), where H and H_2 are direct and staggered (i.e., wavelength 2 lattice sites) magnetic fields. When a fieldlike variable $\tilde{\mathfrak{R}}$ (the ratio of short- to long-range-interaction strengths) is also included, (i.e., $n=4$), Eq. (1.1) indicates that the system has a line of critical points of order $\mathfrak{o}=3$. This is verified in the analysis of Sec. III, where it is shown that in this model there are three surfaces of critical points of order $\mathfrak{o}=2$, meeting at the line of points of order $\mathfrak{o}=3$. The tetracritical point is simply a point on a smooth line of tricritical points; the "tetracritical point" arises because we have chosen a section of the four-dimensional $(H, H_2, T, \tilde{\mathfrak{R}})$ space that is tangent to the line of tricritical points, rather than a section which intersects it.

First we give, in Sec. II, examples of systems exhibiting spaces of critical points of order $\mathfrak{o} \geq 3$, and explain a convenient notation for the phase diagrams of such systems. This leads to an equation equivalent to the Gibbs phase rule.

In Sec. III we explicitly demonstrate the importance of distinguishing between the order of a critical point and the number of lines meeting at a critical point—the former leads to an essential increase in complexity, while the latter does not. To do this, we compare several one-dimensional Ising models with long-range interactions, all of which are exactly soluble.

Special directions at spaces of order $\mathfrak{o}$ are defined in Sec. IV; these are analogous to the strong and weak directions defined by Griffiths and Wheeler.[5] A way of deriving these directions for tricritical points using the renormalization group approach has been pointed out by Riedel and Wegner.[12(c)]

To make the formulation of a scaling hypothesis easier to follow, we give an account in Sec. V of the scaling hypothesis using generalized homogeneous functions and equations invariant under a one-parameter continuous group of transformations at points of order 2.

In Sec. VI we give a full account of the scaling

hypothesis at tricritical points and we include an account of a space of invariant variables as a very useful way to derive "double-power" scaling functions and to plot "crossover surfaces". These predictions have not yet been tested experimentally.

We present the scaling hypothesis at a critical point of arbitrary order in Paper II. The hypothesis is framed as a sequence of operations to be repeated as the hypothesis is formed successively at critical points of decreasing order. The proposed sequence is illustrated by detailed consideration of a system of Ising planes with a variable interplanar interaction.

II. SYSTEMS EXHIBITING CRITICAL POINTS OF ORDER HIGHER THAN TWO: NOTATION FOR SUCH POINTS

Critical points more complex than ordinary $\mathfrak{o}=2$ critical points have been found in many experimental and theoretical systems. Without doubt, the systems exhibiting the richest possibilities are multicomponent fluid mixtures; however, specific examples of critical points of order three or more in these systems have yet to be found.

Systems on which experiments have been analyzed are liquid helium,[10(a)] ammonium chloride,[10(b)] metamagnets,[13] and anisotropic antiferromagnets.[14] In addition, one-dimensional magnetic models provide a rich opportunity for theoretical and numerical investigations. Liquid helium[10(a)] and NH_4Cl[10(b)] provide excellent examples where there exist lines of tricritical points—as discussed in Sec. I. In metamagnets,[13] lines of tricritical points can be generated by introducing transverse fields, and also by introducing a parameter into the Hamiltonian which changes the strength of the interaction. Decreasing the latter causes the tricritical points to converge to a point on the temperature axis; this point is a critical point of order 4 and is treated in more detail below.

An anisotropic antiferromagnet,[14] which exhibits a spin-flop transition, contains a point in its phase diagram where two lines of ordinary critical points intersect a line of first-order transitions. This special point has an order of at least 3, but whether it is 3 or more has yet to be determined.

To be able to discuss phenomena in phase diagrams of any complexity easily, we introduce a notation for spaces of points where several phases coexist and for critical points of arbitrary order $\mathfrak{o}$. Critical spaces are denoted by an abbreviation of the notation CRS of Griffiths and Wheeler[5]; the order of the space will be given a preceding superscript and the dimensionality a subscript, and hence a critical space of order $\mathfrak{o}$ and dimensionality d is written as ${}^{\mathfrak{o}}R_d$. The relation between $\mathfrak{o}$, d, and the total number n of field (or fieldlike) variables, in general, is given by Eq. (1.1), $\mathfrak{o}=n-d$.

Coexistence spaces are designated by an abbreviation of the Griffiths-Wheeler notation CXS but now the number of coexisting phases is given by the preceding superscript; hence the general space of dimension d where p phases coexist is written as ${}^pX_d$. The equation analogous to Eq. (1.1) is

$$p=n-d+1. \tag{2.1}$$

The dimension d of the CXS may be interpreted as the number of "degrees of freedom", f, and for a chemical system n is one greater than the number of components c. Thus (2.1) is similar to the usual statement of the Gibbs phase rule, $p \geq c-f+2$, but it is in a form valid for all the systems considered in this work. The Gibbs phase rule contains an inequality because it refers to any phase diagram, even those in a restricted space of fields. For example, the ${}^2X_2$ in the H-T plane of a simple nearest-neighbor antiferromagnet (with field variables H, H_2, and T) satisfies Eq. (2.1) as an equality providing all three fields are considered, but as an inequality if only T and H are considered. Consideration of other models[8,15] leads us to conclude that Eq. (2.1) is satisfied as an equality (for $T>0$) if and only if a sufficient set of conjugate fields (i.e., conjugate to every possible phase of the system) has been introduced. Thus Eq. (2.1) can be used as a criterion for whether enough conjugate fields have been considered or not. It is noteworthy that Eqs. (1.1) and (2.1) depend only on topologically significant quantities like the dimension of a subspace in the phase diagram, and should therefore be understood as topological statements.

To illustrate the notation and to provide a good example of a system with a phase diagram exhibiting critical spaces of orders 2, 3, and 4, we consider the $d=3$ Ising model with variable interaction strength $\mathfrak{R}J$ between planes of constant z:

$$\mathcal{H}=-\sum_{x,y,z} S_{x,y,z}\left[J(S_{x+1,y,z}+S_{x,y+1,z}) + \mathfrak{R}JS_{x,y,z+1}+H+(-1)^z H_2\right]. \tag{2.2}$$

Here the symbol $S_{x,y,z}$ is the value of the spin on lattice site with coordinates (x, y, z). The variable $\mathfrak{R}$ allows for a variation in the strength of interaction in the z direction. The phase diagram of this model is four dimensional, with fields H, H_2, T, and the fieldlike variable $\mathfrak{R}$. For $\mathfrak{R}>0$, we have a three-dimensional Ising model with "lattice anisotropy," which tends as $\mathfrak{R}\to 0$ to the two-dimensional Ising model. The invariance of the Hamiltonian under the transformation $S_{x,y,z}\to(-1)^z S_{x,y,z}$, $\mathfrak{R}\to-\mathfrak{R}$, $H\to H_2$, and $H_2\to H$ relates the phase diagram for $\mathfrak{R}<0$ to that for $\mathfrak{R}>0$.

For $\mathfrak{R}<0$, Eq. (2.2) describes a metamagnet. The phase diagram of this system is well known and shown in Fig. 1. As $|\mathfrak{R}|$ is decreased, the values of T_t and T_N decrease (unpublished results

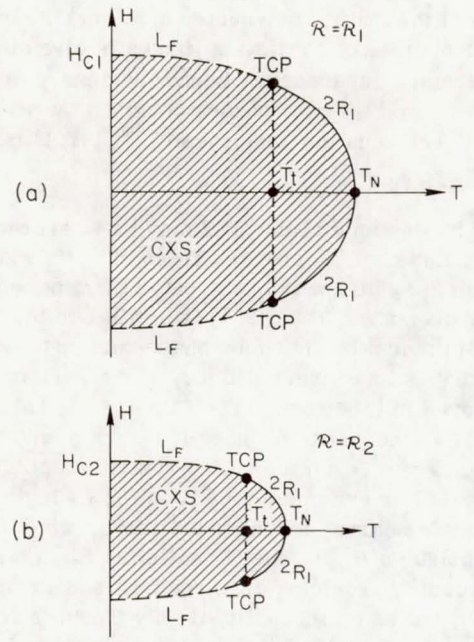

FIG. 1. (a) Phase diagram for $\mathcal{R}=\mathcal{R}_1<0$, in the (H, T) plane $(H'=0)$. The $^2R_1$ terminates at a pair of tricritical points ($^3R_0$) shown as TCP. The $^2X_2$ separating the antiferromagnetic phases A^+A^- is bounded below the tricritical temperature T_t by the lines of first-order transitions L_F, which terminate at a magnetic field value H_{C1}. (b) The same for $\mathcal{R}_2<0$, where $|\mathcal{R}_2|<|\mathcal{R}_1|$. Note that $H_{C2}<H_{C1}$ and that both T_t and T_N have decreased. See also, F. Harbus *et al.*, Ref. 1.

of F. Harbus). This is shown in the Fig. 1(b) as compared to Fig. 1(a).

As $|\mathcal{R}|\to 0$, the behavior of $T_N(\mathcal{R})$ is described by the well-known crossover exponent and symmetry between $\mathcal{R}<0$ and $\mathcal{R}>0$ mentioned above shows that in the plane $H=H_2=0$ there is a reflection symmetry about $\mathcal{R}=0$.

From these considerations we obtain Fig. 2, which is a three-dimensional phase diagram in the $H_2=0$ plane. For $\mathcal{R}<0$ and constant, there is a phase diagram like Fig. 1, and for $\mathcal{R}>0$ the ordinary crossover behavior holds.

The two tricritical points in Fig. 1 become lines of critical points of order 3, $^3R_1$, in Fig. 2. The symmetry of the Hamiltonian shows that there are two additional $^3R_1$ for $\mathcal{R}>0$ at nonzero H_2. The symmetry of the Hamiltonian forces these four lines (tricritical lines) to converge upon a point lying on the temperature axis—a critical point of order 4. On the temperature axis below the $^4R_0$ four phases are in coexistence: it is a $^4X_1$.

The validity of Eqs. (1.1) and (2.1) may be verified and it can also be seen that in this system a $^pX_{d+1}$ is bounded for increasing T by a $^\mathcal{O}R_d$, where $\mathcal{O}=p$. A CXS which in the full phase diagram is a $^pX_{d+1}$ is, when considered in the zero-temperature plane, only a ^pX_d. In the model considered in this section, therefore, a coexistence hypersurface which in the $T=0$ phase diagram is a ^pX_d evolves as the temperature increases into an $^\mathcal{O}R_d$, with $\mathcal{O}=p$. In other words, the space of critical points of order $\mathcal{O}$ is the upper bound (as temperature increases) of a space of points where $\mathcal{O}$ phases coexist.

This is a very significant point and can be understood by examples, and from the following consideration. In a phase diagram, a space where three phases coexist is necessarily a place of intersection of spaces where two phases coexist. The spaces where two phases coexist are bounded from above by spaces of critical points of order two. Therefore, the upper bound of the space where three phases coexist is either the boundary of one critical space of order 2 or the intersection of all three critical spaces of order 2. Because of the symmetry properties holding in the present model, and

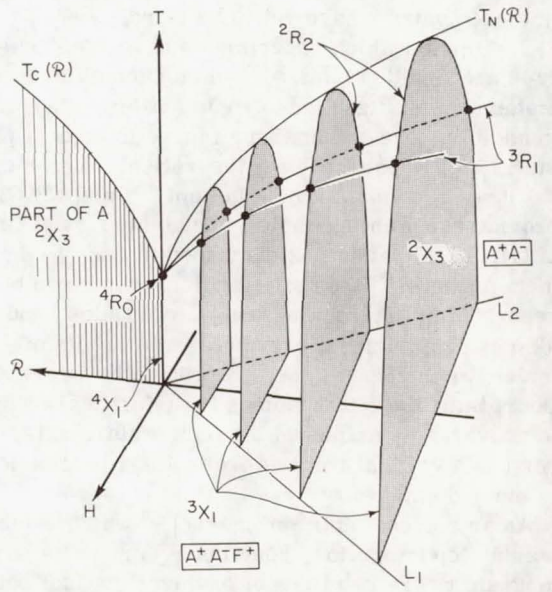

FIG. 2. Phase diagram in the $H_2=0$ hyperplane for Ising model with variable interplanar interaction. For $\mathcal{R}<0$ the $\mathcal{R}=$const sections are similar to Fig. 1. These sections are schematically shaded. As $\mathcal{R}$ varies continuously the ^pX_d and $^\mathcal{O}R_d$ of Fig. 1 become $^pX_{d+1}$ and $^\mathcal{O}R_{d+1}$. Thus the $^2X_2$ of Fig. 1 becomes the interior of the "mountain"; this is a $^2X_3$ separating phases A^+, A^-. The lines L_F where three phases coexist become surfaces of the mountain ($^3X_2$) below the line of tricritical points $^3R_1$ corresponding to TCP of Fig. 1. The $^2R_1$ of Fig. 1 becomes a $^2R_2$, the top of the "mountain" in Fig. 2. The T axis becomes a line of special points where all four phases A^+, A^- and the ferromagnetic F^+ and F^- all coexist; it is a $^4X_0$. The $^3R_1$ meet at the T axis at the end of this line; at a $^4R_0$. The region $\mathcal{R}>0$ appears simpler because it corresponds to the $(H=0)$ section of $\mathcal{R}<0$, and the rest of the mountain occurs at $H_2\neq 0$. See also, F. Harbus *et al.*, Ref. 1.

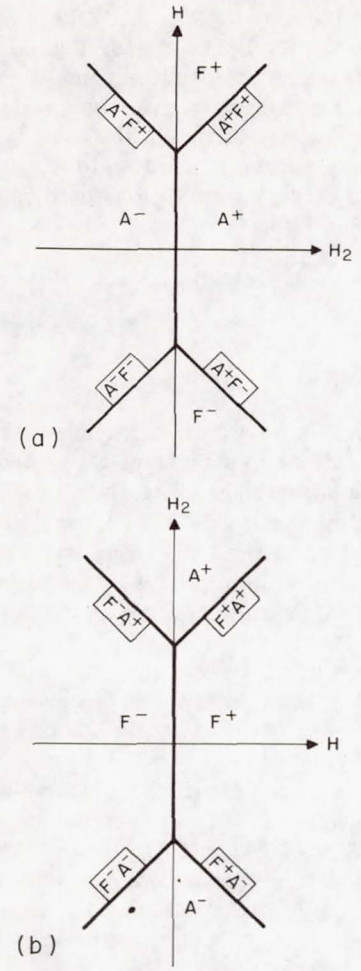

FIG. 3. The phase diagram at $T=0$ of an Ising (a) antiferromagnet and (b) ferromagnet. The phases with spins parallel are indicated by $F^{\pm}$ and with spins antiparallel by $A^{\pm}$. The lines indicate where the various phases are in equilibrium.

also because it is an Ising model, the latter condition holds. An entirely analogous argument can be constructed for critical points of order 4 or more.

It is therefore possible, for Ising models, to make predictions about the relationships between ${}^{\theta}R_d$ in the full phase diagram, by considering the relationships between the ${}^{p}X_d$ in the $T=0$ phase diagram. We will make extensive use of this method in Sec. III.

III. SPACES OF TRICRITICAL POINTS IN ONE-DIMENSIONAL MODELS

There has been much work recently on one-dimensional Ising models possessing a long-range interaction.[8,15] The effect of this interaction is to shift the critical temperature from the value $T=0$ to a nonzero temperature, thereby enabling the critical points to obey scaling laws.

The purpose of this section is to display two models for which critical points of order $\theta=4$ occur; these are both Ising models with long-range interactions.[16]

Models exhibiting a critical point of order 4 also possess a line of points where four phases coexist, as explained at the end of Sec. II. Therefore, a simple method of deciding whether a particular model can possess a critical point of order 4 is to analyze the $T=0$ hyperplane of the phase diagram and see if points where four phases coexist continue to have four distinct phases in equilibrium as T is increased. Cases where this does and does not happen are discussed below.

Before analyzing a case where there is a long-range interaction, let us consider the $T=0$ phase diagram of a one-dimensional Ising antiferromagnet with only nearest-neighbor interactions.

Hamiltonians

The Hamiltonian is given by

$$\mathcal{H}_{SR} = -J_{SR}\sum_{i=1}^{N-1} s_i s_{i+1} - H\sum_{i=1}^{N} s_i - H_2\sum_{i=1}^{N}(-)^{i+1}s_i\,, \tag{3.1}$$

where J_{SR} is the nearest-neighbor (nn) interaction strength and $s_i=\pm 1$ are the Ising spins situated at site i of the chain. When $J_{SR}>0$ the interaction is ferromagnetic and when $J_{SR}<0$ the interaction is antiferromagnetic. H is the magnetic field and H_2 is the staggered magnetic field of wavelength 2 lattice sites. The phase diagram will appear as in Fig. 3(a). Here the four phases $F^{\pm}$, $A^{\pm}$ are defined in Table I: F means ferromagnetic and A means antiferromagnetic. The $T=0$ "critical point" of the

TABLE I. Definitions, energies, and equations of Figs. 3 and 5. Here $\tilde{\mathfrak{K}}\equiv J_{SR}/J_{LR}$; $h\equiv H/J_{LR}$; $h_2\equiv H_2/J_{LR}$. Star means not shown in Fig. 5.

Configuration	Name	Field	Energy
↑ ↑	F^+	H	$E_F=-J_{LR}-J_{SR}-H$
↓ ↓	F^-		
↑ ↓	A^+	H_2	$E_2=+J_{SR}-H_2$
↓ ↑	A^-		

Surface	Equation	Line	Equation
$[F^+,F^-]$	$H=0$	L_1	$h_2=1+2\tilde{\mathfrak{K}};\ h=0$
			$\tilde{\mathfrak{K}}>-\frac{1}{2}$
$[A^+,A^-]$	$H_2=0$	L_2	$h_2=1-2\tilde{\mathfrak{K}};\ h=0$
$[F^+,A^+]^*$	$h-h_2+1+2\tilde{\mathfrak{K}}=0$	L_3	$h=-1-2\tilde{\mathfrak{K}};\ h_2=0$
			$\tilde{\mathfrak{K}}<-\frac{1}{2}$
$[F^-,A^-]^*$	$h-h_2-1+2\tilde{\mathfrak{K}}=0$	L_4	$h=-1+2\tilde{\mathfrak{K}};\ h_2=0$
$[F^+,A^-]^*$	$h+h_2+1+2\tilde{\mathfrak{K}}=0$		
$[F^-,A^+]^*$	$h+h_2-1-2\tilde{\mathfrak{K}}=0$		

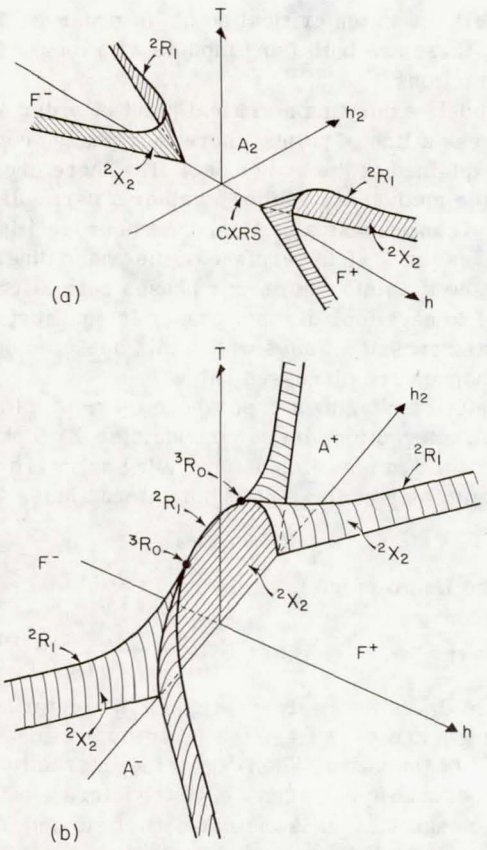

FIG. 4. (a) The extension of Fig. 3(a) into the space with $T>0$ when a weak long-range interaction is included. Note that the $T=0$ plane corresponds to Fig. 3(a), and that the $[A^+, A^-]$ phase boundary only separates phases at $T=0$. The other lines in Fig. 3(a) develop normally, giving coexistence surfaces ($^2X_2$) ending in critical lines $^2R_1$. (b) The phase diagram of the same model but without the antiferromagnetic interaction ($J_{SR}=0$ or $J_{SR}>0$). Now all lines in the $T=0$ plane become coexistence surfaces $^2X_2$. The points of interaction become $^3X_1$ (lines where three phases coexist) and there terminate at two $^3R_0$ (tricritical points).

Ising antiferromagnet becomes a *line* of critical points for nonzero values of magnetic field. This line bifurcates at points where it is energetically more favorable[17] for the system to order ferromagnetically (i.e., with all spins parallel).

Normal scaling laws do not apply to one-dimensional Ising models with short-range interactions, as these display essential singularities at the $T=0$ critical point. Thus the lines in Fig. 3(a) are lines of both critical points and coexistence points. These lines do not have very much in common with either the conventional critical points ^2R_d or the conventional coexistence surfaces ^pX_d that divide up the field space at finite values of temperature. A suitable nomenclature for the lines of Fig. 3(a) might be "coexistence-critical surfaces" and we will denote them by CXRS. A CXRS is necessarily confined to the $T=0$ plane as for Fig. 3(a). Thus we see that the nn Ising antiferromagnet contains five CXRS lines where two phases coexist and two CXRS points where three phases coexist.

Then we introduce in addition to $\mathcal{H}_{SR}$ of Eq. (3.1), a long-range interaction, defined by the Hamiltonian

$$\mathcal{H}_{LR}=-\sum_{i=1}^{N}\sum_{r}J(r)s_i s_{i+r}\,, \tag{3.2a}$$

where

$$J(r)=\lim_{\gamma\to 0} a\gamma e^{-\gamma|r|}\,, \tag{3.2b}$$

the phases $F^{\pm}$ are stabilized at nonzero temperature and continue to show long-range order for $T>0$. The phase diagram is given in Fig. 4(a). The CXRS [on the H axis of Fig. 3(a)] is not stabilized at $T>0$ by the long-range interaction; the two antiferromagnetic phases $A^{\pm}$ coexisting at the CXRS at $T=0$, simply become a single disordered phase for $T>0$.

There are two $^2X_2$ separating the ordered (ferromagnetic) phase from the disordered phase at higher temperatures. These $^2X_2$ end in $^2R_1$ (lines of critical points).

If the dominating nearest-neighbor interactions are ferromagnetic, $J_{SR}>0$, then the situation depicted in Figs. 3(b) and 4(b) results. The two points where three phases coexist at $T=0$ become the end points of two $^3X_1$ lines where three phases coexist. Each $^3X_1$ line terminates at a $^3R_0$ (a tricritical point).

If sufficient care is taken to decide whether a space in the $T=0$ hyperplane is a CXRS or a CXS, then the nature of the extension of the space and its subspaces into $T>0$ can be easily ascertained. The rules exemplified from Figs. 3 and 4 are the following: (i) Two phases which can only be distinguished by a staggered magnetic field coexist on a CXRS. (ii) Such staggered phases give only one phase for $T>0$. (iii) A line where one phase which maintains order for $T>0$ coexists with any other phase is *always* a CXS. (iv) A point where a CXRS meets a CXS has no special properties. It is simply a point on a CXS.

Using these rules we can analyze the model of Nagle and Bonner[8] which includes a long-range interaction, a variable short-range interaction, and the two fields of Fig. 3.[18] A point where four critical lines meet, a tetracritical point, is known for this model. We will show here that this point is a critical point of order 3. Since a variant of this model, which we discuss below, shows a critical point of order 4, it is worth treating the Nagle-Bonner model in some detail first.

Figure 5 depicts the surfaces of coexistence of the four phases in the $T=0$ plane. Definitions and

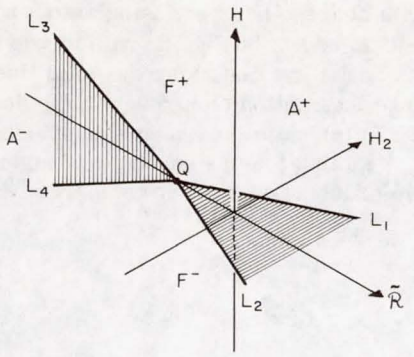

FIG. 5. $T=0$ hyperplane for an Ising model with competing long- and short-range interactions. Here $\tilde{\mathcal{R}} \equiv J_{SR}/J_{LR}$ and H_2 is a staggered magnetic field of wavelength 2 lattice sites. The lines L_1, L_2 are in the $(H_2, \tilde{\mathcal{R}})$ plane, and the lines L_3, L_4 are in the $(H, \tilde{\mathcal{R}})$ plane. The surfaces $[A^-F^+]$, $[A^+F^-]$, $[F^+, A^+]$, $[F^-, A^-]$ have been omitted. The surface $[A^+, A^-]$ is a CXRS, Fig. 3(a) is an (H_2, H) plane for $\tilde{\mathcal{R}} < \tilde{\mathcal{R}}_Q$, while Fig. 3(b) is for $\tilde{\mathcal{R}} > \tilde{\mathcal{R}}_Q$.

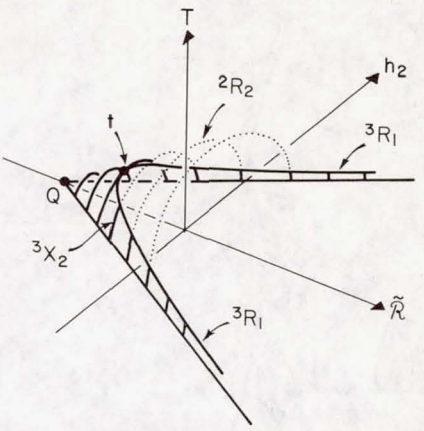

FIG. 6. $H=0$ hyperplane of the model of Fig. 5. The lines L_1, L_2 extend into $T>0$. The lines where three phases coexist in Fig. 4(b) have become a single surface, denoted here by ${}^3X_2$. This terminates at a *line* of tricritical points ${}^3R_1$; the ${}^3R_1$ also bounds a surface of ordinary critical points ${}^2R_2$. The point t where the ${}^3R_1$ intersects the plane $h_2=0$ is the location of the tetracritical point (Ref. 8). The $\tilde{\mathcal{R}}=0$ section of this figure corresponds to the $H=0$ section of Fig. 4(b). Note that at t a surface $\tilde{\mathcal{R}}=$ const is parallel to the ${}^3R_1$.

equations are given in Table I. For clarity four surfaces are omitted; e.g., those that separate the phases $[F^+A^+]$, $[F^-A^-]$, $[F^+A^-]$, and $[F^-A^+]$. As may be seen there are four lines where three phases coexist and a point Q where all four phases coexist. The surface $[A^+A^-]$ bounded by the lines L_3, L_4, is a CXRS since for constant values of $\tilde{\mathcal{R}}$ less than x_Q, the phase diagrams are the same as Figs. 3(a) and 4(a). Here

$$\tilde{\mathcal{R}} \equiv J_{SR}/J_{LR} \ , \tag{3.3}$$

where J_{LR} is the "equivalent neighbor" long-range parameter.[8]

For $T \neq 0$, the two ${}^2X_2$ surfaces $[A^+F^+]$ and $[A^-F^-]$ of Fig. 5 end in a single surface of critical points, and the same is true for the $[A^-F^+]$, $[A^+F^-]$ surfaces; this is shown in Fig. 4(a). The point Q of Fig. 5, where four phases coexist,

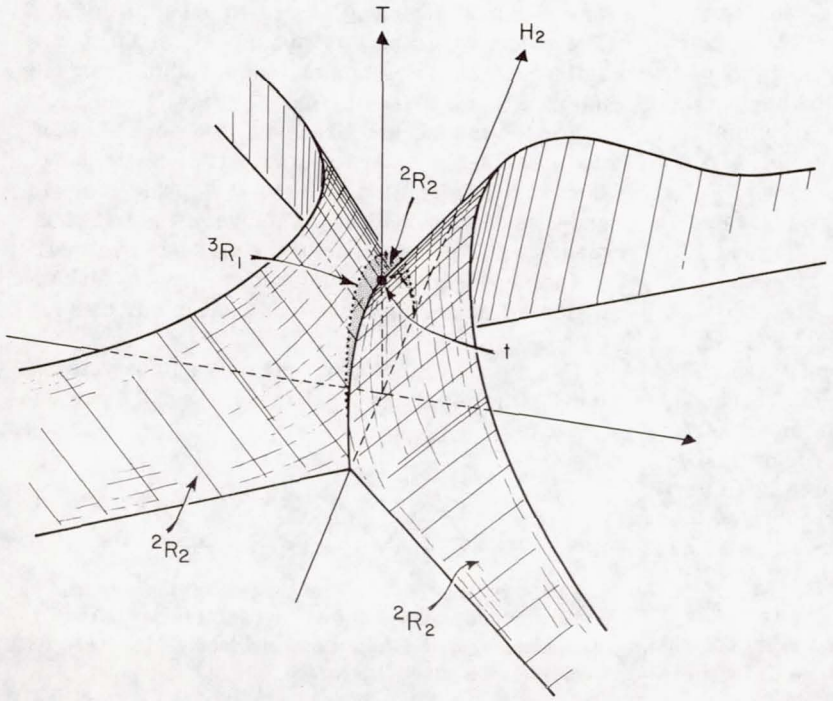

FIG. 7. Ray projection of the four-dimensional space, from Q onto $\tilde{\mathcal{R}}=0$ showing the topology of the surfaces of critical points. The $\tilde{\mathcal{R}}$ axis in Figs. 5 and 6 has become a combination of $\tilde{\mathcal{R}}$ and h. The curved ${}^2R_2$ surfaces are the ends of those surfaces which end on L_3 and L_4 of Fig. 5. The ${}^2R_1$ of Figs. 4(a) and 4(b) are lines in this surface. The flat ${}^2R_2$ shown in the (T, H_2) plane is the surface ${}^2R_2$ of Fig. 6.

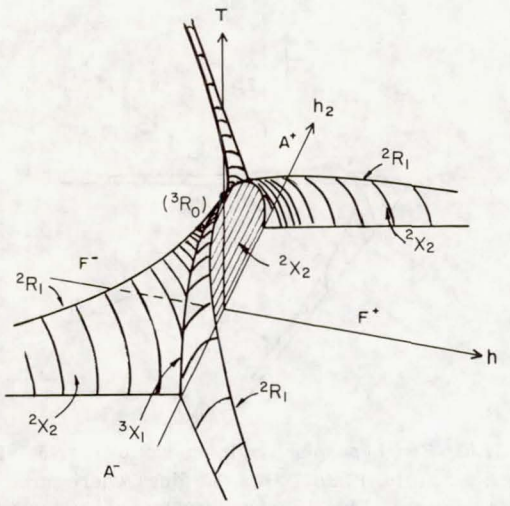

FIG. 8. Plot of the $^2X_2$ and the four lines of critical points meeting at the tetracritical point t. The value of the interaction strength ratio $\tilde{\mathcal{R}} \equiv J_{SR}/J_{LR}$ is equal to its appropriate value $\tilde{\mathcal{R}}_c = 0$.

is seen (Fig. 6) to be only a special point on a $^3X_2$ and Q does *not* give rise to a $^4X_1$ on increasing the temperature. To see this (in the full phase diagram) it is necessary to consider Figs. 4(b) and 5–7. First, in Fig. 4(b), the phase diagram of the system is shown at constant $\tilde{\mathcal{R}} > 0$. The relationship of this figure to Fig. 5 can be understood by looking at the $T = 0$ plane of Fig. 4(b). The lines where two phases coexist are lines on the appropriate surfaces of Fig. 5, and points where lines in the $T = 0$ plane of Fig. 4(b) meet are points on the lines L_1, L_2. Therefore Fig. 4(b) shows that the lines L_1, L_2 do give rise to coexistence surfaces and are lines on $^3X_2$, as shown in Fig. 6. The line of points where three phases coexist ($^3X_1$) in Fig. 5 has become a surface ($^3X_2$) in Fig. 6, and this surface is terminated by a single line of tricritical points, $^3R_1$. Figure 6 shows that the point Q is just a point on a $^3X_2$, and does not generate a $^4X_1$.

The three surfaces of critical points generated by the lines of critical points in Figs. 4(a) and 4(b) are shown schematically in Fig. 7. It can be seen that the two $^3R_0$ (tricritical points) of Fig. 4(b) form a continuous line (a $^3R_1$) bounding three $^2R_2$ (surfaces of critical points). The point t was called a tetracritical point by Nagle and Bonner[8] because, for $\tilde{\mathcal{R}} = \tilde{\mathcal{R}}_c$, *four* lines of critical points meet there (see Fig. 8). However, Figs. 6 and 7 show that t is an indistinguishable point on a smooth line of tricritical points of order 3; this is corroborated by the fact[8] that the exponents at t are the same as at the other tricritical points.

To produce a model where the point Q is stable at higher temperatures demands only a slight change in the structure of the interactions. We draw the linear-chain nearest-neighbor Ising antiferromagnet in the form shown in Fig. 9, with the nn antiferromagnetic interaction along the solid lines, and a long-range interaction along each of the dotted lines. The latter stabilizes each sublattice independently, enabling the system to adopt an antiferromagnetic ordering at nonzero temperature:

$$\mathcal{H}_f = \mathcal{H}_{SR} + \mathcal{H}^0_{LR} + \mathcal{H}^E_{LR}, \tag{3.4a}$$

where

$$\mathcal{H}^{0,E}_{LR} = -\sum_i \sum_r J(2r) s_i s_{i+2r}. \tag{3.4b}$$

Here odd-numbered spins are on the top lattice of Fig. 9 and $\mathcal{H}^0$ has all i odd. Even numbered spins are on the lower lattice and $\mathcal{H}^E$ has all i even; $J(2r)$ is defined by Eq. (3.2b). The point Q is now at the origin and stable for $T > 0$, and we are able to have four $^3X_2$; these are generated by the lines L_1, L_2, L_3, L_4 (of Fig. 5) meeting at the $^4X_1$, generated by Q. The $^3X_2$ should end in a $^3R_1$ (as in Fig. 6) but unlike the system of Fig. 6, the $^3R_1$ all terminate at a $^4R_0$.

This Hamiltonian has an important discrete symmetry which will necessarily be reflected by the phase diagram of the solution of the model. It is given by the operation $s_i \to (-)^i s_i$, $H \to H_2$, $H_2 \to H$, $J_{SR} \to -J_{SR}$. Therefore,

$$G(H, H_2, T; +J_{SR}) = G(H_2, H, T; -J_{SR}).$$

Further, the point Q in Fig. 5 (which is now stable at $T > 0$) is now located at the origin. There are now four $^3R_1$; two of which lie in the $H_2 = 0$ hyperplane for $\tilde{\mathcal{R}} < 0$ and these are symmetrically complemented by two more $^3R_1$ in the $H = 0$ for $\tilde{\mathcal{R}} > 0$. These four $^3R_1$ meet at the T-axis at some finite value of T. This point at which all four $^3R_1$ meet is a $^4R_0$. The phase diagram for this model is the same as that for the Ising model with variable interplanar interaction discussed in Sec. II (Figs. 1 and 2).

For the model just discussed, we were able to make use of the extensive analysis of Nagle and Bonner in conjunction with the $T = 0$ phase diagram, and thus we deduced the structure of the full phase diagram. The existence of the discrete symmetry and the consequent analogy with the model discussed in Sec. II makes us more confident in our conclusions.

For the next model, we use only an analysis of the $T = 0$ phase diagram and we make the extrapola-

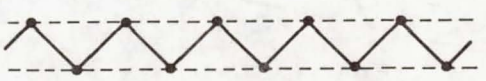

FIG. 9. Modified one-dimensional lattice exhibiting a critical point of order $\mathcal{O} = 4$. The solid lines represent antiferromagnetic interactions and the dashed lines, long-range (ferromagnetic) interactions.

TABLE II. Definitions of spin orderings of phases and fields for an Ising model with two staggered fields.

Spin configuration	Name	Conjugate field
↑ ↑ ↑ ↑	F^+	H
↓ ↓ ↓ ↓	F^-	
↑ ↓ ↑ ↓	A_2^+	H_2
↓ ↑ ↓ ↑	A_2^-	
↑ ↑ ↓ ↓	A_4^+	H_4
↓ ↓ ↑ ↑	A_4^-	
↑ ↑ ↑ ↓	M_a^+	$\frac{1}{2}[H_2+H_4\pm H]$
↑ ↓ ↓ ↓	M_a^-	
↑ ↑ ↓ ↑	M_b^+	$\frac{1}{2}[-H_2+H_4\pm H]$
↓ ↑ ↓ ↓	M_b^-	
↑ ↓ ↑ ↑	M_c^+	$\frac{1}{2}[H_2-H_4\pm H]$
↓ ↓ ↑ ↓	M_c^-	
↓ ↑ ↑ ↑	M_d^+	$\frac{1}{2}[-H_2-H_4\pm H]$
↓ ↓ ↓ ↑	M_d^-	

tions explained at the end of Sec. II. Our knowledge of the extensive occurrence of tricritical points in certain Ising models gives us a reasonable basis from which to predict the existence of a critical point of order 4.

This second model for which an exact solution is fairly readily obtainable[16] is a model with a second staggered magnetic field. It turns out that the analysis for an exact solution is simplest if the staggered field, "H_4," is of wavelength four lattice sites.[19] The Hamiltonian will therefore be given by

$$\mathcal{H}=\mathcal{H}_{SR}+\mathcal{H}_{LR}-H_4\sum_{i=1}^{N}u_i s_i\,, \tag{3.5}$$

where the number $u_i=+1$ for $i=4n+1,\ 4n+2$ and $u_i=-1$ for $i=4n-1,\ 4n$. We define the names of the various phases in Table II and also the phase of H_4 relative to H_2. We give the energies of the phases at $T=0$ in Table III. The phase adopted at $T=0$ is that of lowest energy, and so the problem of finding the phase diagram at $T=0$ is simple. The phases coexist at points where the energies of two different phases are equal, and the most important equalities are given in Table IV(a).

An extended analysis[16] shows that the significant values of $\tilde{\mathfrak{R}}$ are -1, $-\frac{1}{2}$, and 0, and a sequence of phase diagrams can be drawn in the space of variables H, H_2, H_4 for values of $\tilde{\mathfrak{R}}$ greater than, equal to, and less than these numbers. A representative

TABLE III. Energies of various phases at $T=0$ for an Ising model with two staggered fields. Here the long-range energy is given by J_{LR} and the nearest neighbor by J_{SR}. Only the phase + is shown; to get the value of E_{X^-}, reverse the sign of the conjugate field.

Phase	Energy per spin
F^+	$E_F=-J_{LR}-J_{SR}-H$
A_2^+	$E_2=+J_{SR}-H_2$
A_4^+	$E_4=-H_4$
M_a^+	$E_a=-\frac{1}{2}J_{LR}-\frac{1}{2}[+H_2+H_4+H]$
M_b^+	$E_b=-\frac{1}{2}J_{LR}-\frac{1}{2}[-H_2+H_4+H]$
M_c^+	$E_c=-\frac{1}{2}J_{LR}-\frac{1}{2}[+H_2-H_4+H]$
M_d^+	$E_d=-\frac{1}{2}J_{LR}-\frac{1}{2}[-H_2-H_4+H]$

TABLE IV. (a) Equations of planes and lines in Fig. 10 for an Ising model with two staggered fields. Here we have divided all energies through by J_{LR} and defined $h_i=H_i/J_{LR}$, $\tilde{\mathfrak{R}}\equiv J_{SR}/J_{LR}$. In Fig. 10, $\tilde{\mathfrak{R}}$ is positive, e.g., $\frac{1}{2}$ or ∞. Star means that the plane was omitted from Fig. 10. Double star indicates that this is only a line: (b) Equations of lines in Fig. 10.

(a)

Phase boundary	Equation	Region
$[F^+,F^-]$	$h=0$	
$[M_a^+,M_a^-]$	$h=0$	
$[F^+,M_a^+]^*$	$h_2+h_4-1-2\tilde{\mathfrak{R}}=h$	$h,h_2,h_4>0$
$[F^+,M_b^+]^*$	$h_2-h_4+1+2\tilde{\mathfrak{R}}=-h$	$h,h_4>0,\ h_2<0$
$[F^+,A_4^+]$	$h_4-1-\tilde{\mathfrak{R}}=h$	$h_4>0,\ h>0$
$[A_4^+,M_a^+]$	$h_4-h_2-1=h$	$h,h_4>0,\ h_2>0$
$[A_4^+,M_b^+]$	$h_4+h_2-1=h$	$h,h_4>0,\ h_2<0$
$[F^+,A_2^+]^{**}$	$h_2-1-2\tilde{\mathfrak{R}}=h$	$h,h_2>0;\ h_4=0$
$[A_2^+,M_a^+]$	$h_2-h_4-1-2\tilde{\mathfrak{R}}=h$	$h,h_2,h_4>0$

(b)

Line	Intersection	Equation
$[4:F^+F^-M_a^+M_a^-]$	$[F^+,F^-][M_a^+,M_a^-]$	$h_2+h_4=1+2\tilde{\mathfrak{R}}$
	$[F^+,M_a^+],[F^-,M_a^-]$	$h=0$
$[3:F^+F^-A_4^+]$	$[F^+,F^-][F^+,A_4^+]$	$h_4=1+\tilde{\mathfrak{R}}$
	$[F^-,A_4^+]^*$	$h=0$
$[3:F^+M_a^+A_4^+]$	$[F^+,A_4^+][A_4^+,M_a^+]$	$h_4-h=1+\tilde{\mathfrak{R}}$
	$[F^+,M_a^+]^*$	$h_2=\tilde{\mathfrak{R}}$
$[3:F^+M_b^+A_4^+]$	$[F^+A_4^+][A_4^+,M_b^+]$	$h_4-h=1+\tilde{\mathfrak{R}}$
	$[F^+,M_b^+]^*$	$h_2=-\tilde{\mathfrak{R}}$
$[4:F^+M_a^+M_c^+A_2^+]$	$[F^+,M_a^+]^*[F^+M_c^+]^*$	$h_2-h=1+2\tilde{\mathfrak{R}}$
	$[M_a^+,A_2^+][M_c^+,A_2^+]$	$h_4=0$
$[3:M_a^+,M_a^-,A_4^+]$	$[M_a^+,M_a^-][M_a^+,A_4^+]$	$h_4-h_2=1$
	$[M_a^-,A_4^+]^*$	$h=0$
$[3:M_a^+,M_a^-,A_2^+]$	$[M_a^+,M_a^-],[M_a^+,A_2^+]$	$h_2-h_4=1+2\tilde{\mathfrak{R}}$
	$[M_a^+,A_2^-]$	$h=0$

diagram is shown in Fig. 10 for which the equations of the lines are given in Table IV(b). Some of the surfaces and lines of Fig. 10 are labeled and the reader can discover the labels for the rest by reference to Tables IV(a) and IV(b).

There are several important points about this model: firstly, the "mixed" phases $M_a^{\pm}$ are distinguished at $T=0$ as $M_a^+ \cdots M_d^+$ but above absolute zero there are only phases M^+ and M^-. The phases $M^{\pm}$ are stable for $T>0$ because they contain a long-range contribution to their energy; thus all the lines and surfaces on Fig. 10 will survive at $T>0$ because they separate phases stabilized by the long-range interaction from phases ($A_2^{\pm}$, $A_4^{\pm}$) stable only at $T=0$.

There are several points where many phases coexist. In particular, at the point P_2 the phases $F^{\pm}$, $M_a^{\pm}$, and A_4^+ coexist. The lines where three or more phases coexist which meet at P_2 are all stable at $T>0$ and so should end in tricritical points. Thus P_2 will give rise to a line of points where five phases coexist; this line ends at a critical point of order 4. Other points in Fig. 10 (viz., P_1) are much more complex in structure and will not be discussed here. The object of introducing the model given by Eq. (3.5) was to show a critical point of order four and this, at least, we have done.

In this section we have shown that it is reasonably easy to find model systems which are soluble and which show critical points of order 4 or more. The analysis of the two models suggested was omitted for the sake of brevity, and will be given in future work.[16]

IV. SPECIAL DIRECTIONS AT CRITICAL SPACES OF ORDER ∞: A SET OF "CANONICAL DIRECTIONS"

In order to properly formulate the scaling hypothesis for multicomponent systems, it is important to choose the proper independent variables. It is this problem that is treated in the present section. We shall argue that the considerations that Griffiths and Wheeler[5] applied to their discussion of second-order critical spaces (${}^2R_d$) can be extended to spaces of higher order in a natural and straightforward fashion.

A ${}^2X_d$ is, by definition, a hypersurface where two phases coexist; it necessarily divides the total space of n field variables locally into two regions and is therefore of dimension $d=n-1$, where n is the total number of truly intensive or "field" variables.[5] A ${}^2R_d$ (a simple second-order critical space) is the boundary of a ${}^2X_{n-1}$ and is therefore of dimension $d=n-2$.

At a ${}^2R_{n-2}$ there are $n-2$ directions parametrizing the critical space. The two remaining directions are of significance for the generalized scaling hypothesis. Directions not locally parallel to the ${}^2X_{n-1}$ (coexistence surface) we call strong directions, and directions locally parallel to the ${}^2X_{n-1}$ but not in the ${}^2R_{n-2}$ we call weak directions. The strong and weak directions will be called directions of type 1 and 2, respectively; this terminology is useful in Sec. VI and in Paper II, where the appropriate generalization to critical spaces of order larger than 2 is made. Examples are given in Figs. 11 and 12.

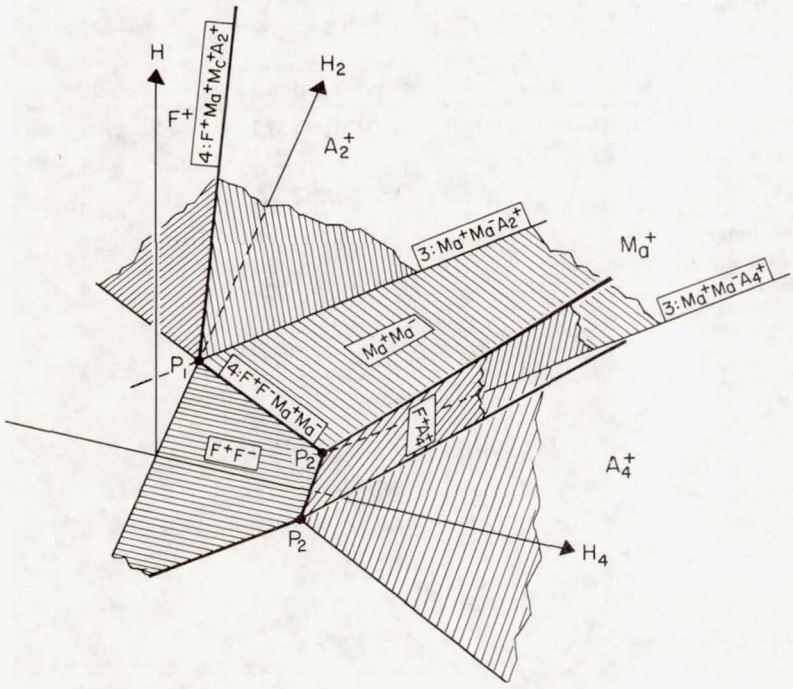

FIG. 10. Coexistence surfaces in the $T=0$ phase for a system with a long-range interaction, a nn interaction, magnetic field h, and staggered magnetic fields h_2, h_4 of wavelengths 2 and 4. Here the short-range interaction is also ferromagnetic. The surfaces are labeled by the phases in coexistence. The surface $[F^+, M_a^+]$ is omitted. The lines are labeled by the three or four phases in coexistence there. At the point P_2, five phases coexist; at P_1, seven phases coexist. The phases F and M are stable above $T=0$.

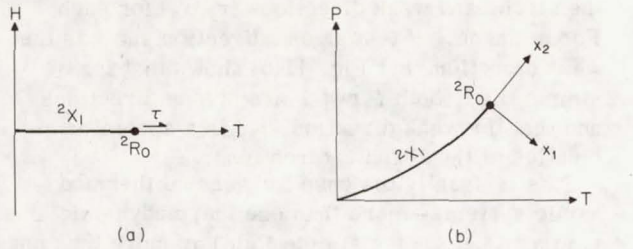

FIG. 11. (a) Phase transition for a ferromagnetic system. The coexistence surface $^2X_1$ ends at a critical point $^2R_0$. The strong direction H and weak direction τ are defined at the $^2R_0$. (b) Phase transition for a simple fluid. The coexistence surface $^2X_1$ ends at a critical point $^2R_0$. The strong direction x_1 and weak direction x_2 are indicated; both P and T directions are strong.

Next, we introduce the concept of a direction of type 3. In the examples discussed, the critical spaces of order 3 have a dimension of $d=n-3$. If one approaches a particular tricritical point along a line of critical points of order 2, then in addition to the two directions singled out by the $^2R_1$ (and its associated $^2X_2$), there is a third direction of significance for scaling. This direction, which we call a direction of type 3, is a direction tangent to the line of critical points (Figs. 12 and 14). This concept is easily generalized to $n>3$ for the case mentioned above, for which the dimension d of the space of critical points of order three is indeed given by $d=n-3$. In this case directions of type 3 are those directions which are neither strong nor weak for a particular $^2R_{n-2}$ bounded by the $^3R_{n-3}$, nor are they locally tangent to the space of critical points of

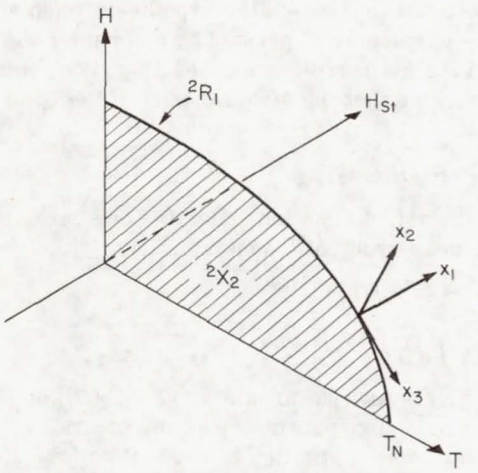

FIG. 12. Phase transition for an antiferromagnet. The coexistence surface $^2X_2$ lies in the H, T plane and the line of critical points $^2R_1$ bounds it. The strong direction is everywhere H_{st}, the staggered magnetic field. The weak direction may be either H or T, *except* at the Néel point where it can only be T. The independent direction x_0 lies in the $^2R_1$.

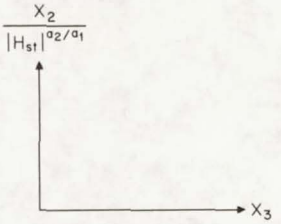

FIG. 13. Invariant space for an antiferromagnet. (x_3 is merely a parameter.)

third order.

In the similar cases where Eq. (1.1) holds as an equality, one can generalize the above concepts to define directions of types 1, 2, ..., $\mathfrak{o}$, and these are important in applying the scaling hypothesis to critical spaces of order $\mathfrak{o}$. Specifically, since a critical point of order $\mathfrak{o}$ is, by definition, a particular point on a "line" of critical points of order $\mathfrak{o}-1$, the generalization follows by analogy with the case treated above. The directions of types 1 through $\mathfrak{o}$ are of great importance because they are used as the independent variables of the Gibbs function when the scaling hypothesis is made. Accordingly, they will be referred to in later sections as the "principal directions of scaling."

Thus, to set up a coordinate system at a $^{\mathfrak{o}}R_{n-\mathfrak{o}}$ (a general critical space of order $\mathfrak{o}$), a set of critical spaces $^jR_{n-j}$ of orders $j=2, 3, \ldots, \mathfrak{o}$ must be selected. This set of spaces must satisfy an inclusion principle: $^{\mathfrak{o}}R_{n-\mathfrak{o}} \subset {}^iR_{n-i} \subset {}^jR_{n-j}$ for $j<i<\mathfrak{o}$. The directions of types 1, 2, ..., $\mathfrak{o}$ are then sequentially defined. Here the inclusion symbol $\subset$ means not only that the $^iR_{n-i}$ is also part of a $^jR_{n-j}$ (for $i>j$) but also that the $^iR_{n-i}$ can be reached as a limiting point or boundary of the $^jR_{n-j}$.

The hierarchy of spaces $^jR_{n-j}$ is not unique, and the large number of choices available presents an apparent problem because many more than $\mathfrak{o}$ linearly independent vector directions are definable. For the $\mathfrak{o}=3$ example of Fig. 14 the $^3R_0$ can be approached along any of the three $^2R_1$, and each of these three "critical lines" (with its associated $^2X_2$) defines a set of directions of types 1, 2, and 3. This apparent problem is resolved by the generalized scaling hypothesis, because the shape of each critical space of order j ($j<\mathfrak{o}$) near the $^{\mathfrak{o}}R_d$ is constrained by the scaling hypothesis so that all the different directions end up mutually consistent. Accordingly, we now turn our attention to the scaling hypothesis, making it firstly in Sec. V for simple systems ($n=2$), for $n=3$ systems with a $^3R_0$ (tricritical point) in Sec. VI, and in Paper II for a $^{\mathfrak{o}}R_{n-\mathfrak{o}}$ (a general critical space of order $\mathfrak{o}$).[20]

V. INVARIANT THEOREMS OF ONE-PARAMETER CONTINUOUS GROUPS; APPLICATION TO THE SCALING HYPOTHESIS FOR CRITICAL SPACES OF ORDER 2

The scaling hypothesis for a simple system with two independent field variables can be made in

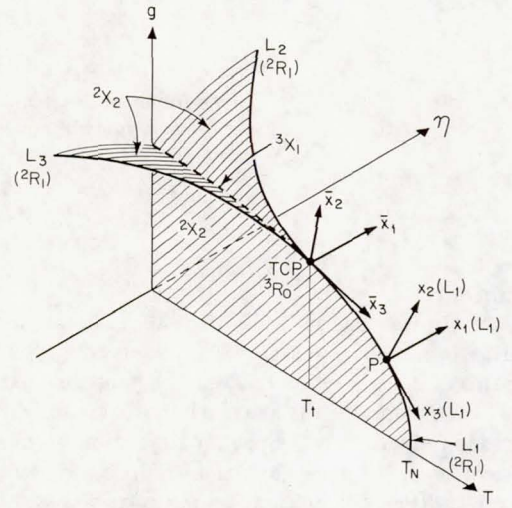

FIG. 14. Phase diagram for a metamagnet. The three $^2X_2$ end in lines of critical points $^2R_1$. These lines intersect at the tricritical point $^3R_0$. At a point P on the line L_1, a triad of strong, weak, and parallel (to L_1) directions is shown. This triad attains the limiting orientation $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ at the tricritical point $^3R_0$.

several essentially equivalent fashions. One statement is that the singular part of the Gibbs potential is asymptotically a generalized homogeneous function (GHF) of the appropriate variables.[21] For the simple magnet, this statement takes the form that there exist two numbers a_H, a_τ (called "scaling powers") such that for all positive λ,

$$G(\lambda^{a_H}H, \lambda^{a_\tau}\tau) = \lambda G(H, \tau), \tag{5.1}$$

where H is the magnetic field and $\tau \equiv T - T_c$. Using Eq. (5.1), one can express all possible thermodynamic exponents in terms of (a_H, a_τ).[21]

This form of the scaling hypothesis implies that the singular part of the Gibbs potential

$$G = F(H, \tau) \tag{5.2}$$

is an invariant equation under the "scaling" transformation defined by

$$G' \equiv \lambda G, \tag{5.3a}$$

$$H' \equiv \lambda^{a_H}H;\ \tau' \equiv \lambda^{a_\tau}\tau, \tag{5.3b}$$

such that $G' = F(H', \tau')$. The transformations defined by Eq. (5.3) form a one-parameter group[22] $\mathcal{G}_F$ and the scaling hypothesis may be restated in the following fashion: (5.2) is an invariant equation under the group of transformations $\mathcal{G}_F$ of (5.3).

For a simple magnet the thermodynamic axes are parallel to the directions used in the scaling hypothesis (5.1) (the "principal axes of scaling"). In other systems, this is not always so. In Fig. 11 we contrast the phase diagrams of a simple fluid and a simple magnet, and show the orientations of the strong and weak directions (x_1, x_2) for each. For a magnet, H is a strong direction and τ is the weak direction, but Fig. 11(b) shows that for a simple fluid, both P *and* T are strong directions and that the weak direction, x_2, is a special combination of the P and T directions.

This is usually the case for general thermodynamic systems—more than one thermodynamic axis is strong (as for the simple fluid) or more than one axis is weak (as for an antiferromagnet, see Fig. 12).

The scaling hypothesis at a $^2R_{n-2}$ for a general system can be made by choosing the strong and weak directions as the principal axes of scaling. Then the statement is that the singular part of the Gibbs function

$$G = \mathcal{F}(x_1, x_2; \ldots, x_n) \tag{5.4}$$

is an invariant equation under the one-parameter[23] continuous group of transformations $\mathcal{G}$:

$$\mathcal{G}\begin{cases} G' = \lambda G, & \text{(5.5a)} \\ x_i' = \lambda^{a_i}x_i, \quad i = 1, 2, \ldots, n, \quad [a_i = 0,\ i > 2]. & \text{(5.5b)} \end{cases}$$

Equation (5.5) is defined for all positive λ; the a_i are called scaling powers. This statement is equivalent to the scaling hypothesis

$$G(\lambda^{a_1}x_1, \lambda^{a_2}x_2; x_3 \cdots x_n) = \lambda G(x_1, x_2; \cdots x_n). \tag{5.6}$$

For future reference, we will use a superscript s to denote the subgroup generated by the transformations of the independent variables x_i $(i = 1, 2, \ldots, n)$. Thus we may generally define the full group by two equations, where the second denotes the subgroup $\mathcal{G}^s$.

To illustrate the scaling hypothesis when an inactive parameter is present, we consider the example of the antiferromagnet (see Fig. 12). Here we hypothesize that the singular part of the Gibbs potential

$$G = \mathcal{F}_A(H_{st}, x_2, x_3) \tag{5.7}$$

is an invariant equation under the one parameter continuous group of transformations $\mathcal{G}_A$:

$$\mathcal{G}_A\begin{cases} G' = \lambda G, \\ H_{st}' = \lambda^{a_1}H_{st},\ x_2' = \lambda^{a_2}x_2,\ x_3' = x_3, \end{cases} \tag{5.8}$$

where x_2 is a weak direction, $[T - T_c(H)]$ or $[H - H_c(T)]$, and x_3 parametrizes the position of the critical point on the line of critical points. Equivalently G satisfies a GHF equation[24]

$$G(\lambda^{a_1}H_{st}, \lambda^{a_2}x_2; x_3) = \lambda G(H_{st}, x_2; x_3). \tag{5.9}$$

Here H_{st} denotes the staggered magnetic field.

Scaling functions for ferromagnets and antiferromagnets can be obtained the usual way[21] from

Eqs. (5.1) and (5.9), respectively. For example, by setting $\lambda \equiv 1/|H|^{1/a_H}$ in Eq. (5.1), we obtain[25]

$$G(1, \tau/|H|^{a_\tau/a_H}) = |H|^{-1/a_H} G(H, \tau). \tag{5.10}$$

The function $G(H, \tau)$ can be plotted as a curve in a two-dimensional plane with coordinates specified by $G/|H|^{1/a_H}$ and $\tau/|H|^{a_\tau/a_H}$. We note that both $G/|H|^{1/a_H}$ and $\tau/|H|^{a_\tau/a_H}$ are absolute invariants of the group $\mathcal{G}_F$ defined by Eq. (5.3), i.e.,

$$G_H \equiv \frac{G'}{|H'|^{1/a_H}} = \frac{G}{|H|^{1/a_H}}, \tag{5.11a}$$

$$\tau_H \equiv \frac{\tau'}{|H'|^{a_\tau/a_H}} = \frac{\tau}{|H|^{a_\tau/a_H}}, \tag{5.11b}$$

where the second equalities in (5.11) follow from (5.3).

Similarly, by setting $\lambda \equiv 1/|H_{st}|^{1/a_1}$, Eq. (5.9) may be written

$$G(1, x_2/|H_{st}|^{a_2/a_1}; x_3) = |H_{st}|^{-1/a_1} G(H_{st}, x_2; x_3), \tag{5.12}$$

and $G/|H_{st}|^{1/a_1}$, $x_2/|H_{st}|^{a_2/a_1}$, x_3 are absolute invariants of the group $\mathcal{G}_A$. In addition, the last two quantities are absolute invariants of the subgroup $\mathcal{G}_A^s$.

To derive exponents and scaling laws for the antiferromagnet, the procedures developed in Ref. 21 can be simply applied to Eq. (5.9). At points where the critical line is not parallel to the T axis, it is easily shown that

$$\beta = (1 - a_1)/a_2, \tag{5.13a}$$

$$1/\delta = (1 - a_1)/a_1, \tag{5.13b}$$

$$-\gamma = (1 - 2a_1)/a_2, \tag{5.13c}$$

$$-\alpha = (1 - 2a_2)/a_2, \tag{5.13d}$$

where the order parameter M_{st} tends to zero when the critical line is approached in a weak direction with exponent β,

$$M_{st} \propto |T - T_c(H)|^{\beta}, \tag{5.14a}$$

and with exponent $1/\delta$ when it is approached in a strong direction

$$M_{st} \propto H_{st}^{1/\delta}. \tag{5.14b}$$

The staggered susceptibility $\chi_{st} = \partial M_{st}/\partial H_{st}$ diverges with exponent γ,

$$\chi_{st} \propto |T - T_c(H)|^{-\gamma}, \tag{5.14c}$$

and the specific heat at constant order parameter diverges with exponent α:

$$C_{M_{st}} \propto |T - T_c(H)|^{-\alpha}. \tag{5.14d}$$

The last two exponents γ and α refer to weak directions of approach (in the plane $H_{st} = 0$) to the critical line.[26]

In the same way, exponents for the ordinary magnetization M, and ordinary susceptibility $\chi = \partial M/\partial H$ can be derived. It turns out on using Eq. (5.13d) that

$$M - M_c(H) \propto |T - T_c(H)|^{(1-\alpha)}, \tag{5.15a}$$

$$\chi \propto |T - T_c(H)|^{-\alpha}. \tag{5.15b}$$

From Eqs. (5.13a)–(5.13d) the usual scaling law equalities can be derived by eliminating the scaling powers a_1 and a_2:

$$\alpha + 2\beta + \gamma = 2, \qquad \beta(\delta - 1) = \gamma, \quad \beta(\delta + 1) = 2 - \alpha.$$

These results are also obtainable from two simple group invariant theorems which we shall find particularly useful in making the scaling hypothesis for critical subspaces of higher order.

Theorem 1. Consider a one-parameter continuous group of transformations:

$$\mathcal{G}\begin{cases} x_0' = f(\lambda | x_0), & (5.16a) \\ x_i' = f_i(\lambda | x_1, x_2, \ldots, x_n), \quad i = 1, 2, \ldots n. & (5.16b) \end{cases}$$

There exist n functionally independent absolute invariants of the x_i $(i = 0, 1, \ldots, n)$. This theorem is proved in Appendix A.

For our applications, we shall choose the invariants, denoted by y_i $(i = 0, 1, \ldots, n-1)$, such that, for $i = 0$,

$$\frac{\partial y_0}{\partial x_0} \neq 0, \tag{5.17a}$$

and for $i > 0$,

$$(y_1, y_2, \ldots, y_{n-1}) \tag{5.17b}$$

are the $n-1$ functionally independent absolute invariants of the subgroup $\mathcal{G}^s$.

For the simple ferromagnet, $G/|H|^{1/a_H}$ is an absolute invariant under $\mathcal{G}_F$ [see Eq. (5.3)] and $\tau/|H|^{a_\tau/a_H}$ is the functionally independent absolute invariant of (H, τ) under the transformation $\mathcal{G}_F^s$. For the antiferromagnet, $G/|H_{st}|^{1/a_1}$ is an absolute invariant under $\mathcal{G}_A$, and $(x_2/|H_{st}|^{a_2/a_1}, x_3)$ are the *two* functionally independent absolute invariants under $\mathcal{G}_A^s$.

Theorem 2. If

$$x_0 = X_0(x_1, x_2, \ldots, x_n) \tag{5.18}$$

is an invariant equation under $\mathcal{G}$ defined by (5.16) (i.e., if X_0 is a GHF), then it can be expressed as

$$y_0 = Y_0(y_1, y_2, \ldots, y_{n-1}), \tag{5.19}$$

where the y_i $(i = 0, 1, \ldots, n-1)$ form a set of functionally independent absolute invariants of $\mathcal{G}$ given by (5.16), and satisfying (5.17).

The proof of this theorem is given in Appendix B. Equations (5.10) and (5.12) are simple applications of these theorems. The usefulness of theorems 1 and 2 will be apparent in Sec. VI.

VI. SCALING HYPOTHESIS FOR TRICRITICAL POINTS

Before making the scaling hypothesis and examining its consequences for a critical space of arbitrary order $\mathfrak{o}$, we will make it for the special case of a ${}^3R_0$ (tricritical point) for which there are three fields available ($n=3$). This will clarify both notation and concepts, and make passage to the general case more painless.

As was shown in Sec. IV, it is possible at a critical point to select strong and weak directions (directions of types 1 and 2). These we call x_1 and x_2 as before. In a space of total dimension three, the critical subspaces terminating at a ${}^3R_0$ are all ${}^2R_1$. At a point on a ${}^2R_1$ we may select a direction tangent to the line. Thus as one approaches the ${}^3R_0$ along a given ${}^2R_1$, the directions of types 1, 2, and 3 are uniquely defined (see Fig. 14).

Since three critical lines meet at a ${}^3R_0$, three "rival" coordinate systems exist at the point ${}^3R_0$. A scaling hypothesis cannot be made at the tricritical point unless a unique coordinate system can be defined, and this represents an apparent obstacle.

The solution of this problem is somewhat subtle, and the full details have been given in a previous paper.[7(b)] The basic idea is that a scaling hypothesis at the ${}^3R_0$ determines the general shape of a line of critical points near the ${}^3R_0$; thus a scaling hypothesis made in a coordinate system defined by one line will restrict the shapes of the other two lines meeting it (at the tricritical point).

The coordinate systems defined with reference to the other two lines are consistent in the sense that we could have selected any line first to make the scaling hypothesis and we would have obtained the same final result.

To set up a coordinate system in which to make a scaling hypothesis at the ${}^3R_0$, we choose a point P on one of the critical lines (say L_1) and we set up a triad of directions $x_i(L_1)$. Two of these directions are of types 1 and 2, while the third is tangent to the ${}^2R_1$. The coordinate system at the tricritical point is now defined to be

$$\bar{x}_i \equiv \lim_{P \to {}^3R_0} x_i(P) \tag{6.1}$$

(see Fig. 14). The direction $\bar{x}_3$ is of type 3. The bars are used in order that the present notation be consistent with that of Ref. 7(b).

We now introduce a scaling parameter λ ($\lambda > 0$) and make the scaling hypothesis that the singular part of the Gibbs potential is a GHF, i.e.,

$$G(\lambda^{\bar{a}_1}\bar{x}_1, \lambda^{\bar{a}_2}\bar{x}_2, \lambda^{\bar{a}_3}\bar{x}_3) = \lambda G(\bar{x}_1, \bar{x}_2, \bar{x}_3), \tag{6.2}$$

where $(\bar{a}_1, \bar{a}_2, \bar{a}_3)$ are the "tricritical scaling powers". Equation (6.2) is equivalent to the statement that $G = \mathfrak{F}(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ is an invariant equation under the one-parameter continuous group of transformations

$$\mathcal{G}_3 \begin{cases} G' \equiv \lambda G, & \text{(6.3a)} \\ \bar{x}_i' \equiv \lambda^{\bar{a}_i}\bar{x}_i, \quad i = 1, 2, 3. & \text{(6.3b)} \end{cases}$$

According to theorem 1 of Sec. V, the hypothesized invariance property of (6.1) under $\mathcal{G}_3$ implies that there exists a basis set of functionally independent absolute invariants of $\mathcal{G}_3$. We adopt a canonical form for the invariants y_i by scaling $\bar{x}_i$ with respect to the tangent variable—here $\bar{x}_3$—as follows:

$$y_0 \equiv \frac{\bar{x}_0}{\bar{x}_3^{1/\bar{a}_3}}, \quad y_1 \equiv \frac{\bar{x}_1}{\bar{x}_3^{\bar{a}_1/\bar{a}_3}}, \quad y_2 \equiv \frac{\bar{x}_2}{\bar{x}_3^{\bar{a}_2/\bar{a}_3}}, \tag{6.4}$$

with $\bar{x}_0 \equiv G$. Thus, theorem 2 of Sec. V states that $G(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ may be expressed as[27]

$$y_0 = F_2(y_1, y_2). \tag{6.5}$$

Since the variables y_1 and y_2 forms a basis set of functionally independent absolute invariants of the scaling field variables $\bar{x}_1$, $\bar{x}_2$, and $\bar{x}_3$ of the group of transformations $\mathcal{G}_3^s$, any point (k_1, k_2) in the two-dimensional space (y_1, y_2) gives rise to an invariant curve of points in the three-dimensional space $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$. That is, the point given by

$$y_i = k_i, \quad i = 1, 2 \tag{6.6}$$

corresponds to a line in the x_i space that may be conveniently parametrized by

$$(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (k_1\lambda^{\bar{a}_1}, k_2\lambda^{\bar{a}_2}, \lambda^{\bar{a}_3}), \tag{6.7}$$

where λ is an arbitrary parameter (see Fig. 15). In particular the lines of critical points L_j converging on the tricritical point can be expressed in the form of Eq. (6.7), since according to the scaling hypothesis (6.2) they must be invariant under the group of symmetries $\mathcal{G}_3$ of (6.3).

Previously[7(b)] we have derived Eq. (6.7) directly from Eq. (6.2) and demonstrated that if the scaling powers $\bar{a}_i$ are all different, the curves parametrized by Eq. (6.7) end up parallel to the axis $\bar{x}_i$ corresponding to the minimum $\bar{a}_i$ (unless $k_i = 0$). Although the direction of type 3 defined for one line is not necessarily parallel to the direction of type 3 defined for another line, it will at least be parallel to *some* member of the triad defined for that other line. Thus all choices of scaling directions will be mutually consistent![7(b),28]

Along a critical line ${}^2R_1$, the conventional scaling hypothesis is normally stated in terms of a GHF equation of the form

$$G(\mu^{a_1}x_1, \mu^{a_2}x_2; x_3) = \mu G(x_1, x_2; x_3), \tag{6.8}$$

where x_3 is a parameter and does not scale. Near the ${}^3R_0$, however, the shape of the critical line is determined by Eq. (6.7). A ${}^2R_1$ near the ${}^3R_0$ maps into a point (k_1, k_2) in the $y_1 - y_2$ plane given by Eq. (6.6). Furthermore, the value of $y_0 \equiv \bar{x}_0/\bar{x}_3^{1/\bar{a}_3}$

$= G/\bar{x}_3^{1/\bar{a}_3}$ changes only if y_1 and/or y_2 changes. It is therefore more proper to make a precise scaling hypothesis about the critical line ${}^2R_1$ near the ${}^3R_0$ making use of the variables (y_1, y_2).

If we adopt the strong requirement that a point in one phase remains in that phase under the scale transformation $\mathcal{G}_3$, the CXS surfaces become lines in the y_1-y_2 plane. Hence, it is possible to choose the principal directions of scaling for the ${}^2R_{n-2}$ as linear combinations of the variables y_1, y_2.

Because the scaled variables must be zero at the critical line we consider the variables

$$\hat{y}_1 \equiv y_1 - k_1 , \tag{6.9a}$$

$$\hat{y}_2 \equiv y_2 - k_2 . \tag{6.9b}$$

For the line L_1 of Fig. 15, $k_1 = 0$, the coordinate y_1, is everywhere strong and $(y_2 + k)$ is weak. For L_2 and L_3, the weak direction is parallel to the CXS mapped in the $y_1 - y_2$ plane, and both $\hat{y}_1$, $\hat{y}_2$ are strong directions unless the wings end up parallel to one axis.

We therefore define linear combinations of the variables $(y_i - k_i)$, which give the weak and strong directions (they are of necessity also absolute invariants of the group $\mathcal{G}_3$):

$$\tilde{y}_i \equiv \sum_{j=1}^{2} \bar{R}_{ij}(y_j - k_j) , \tag{6.10a}$$

where $\bar{R}_{ij}$ is a "rotation matrix". Defining

$$\tilde{y}_0 \equiv y_0 , \tag{6.10b}$$

we hypothesize that along a ${}^2R_1$ near the ${}^3R_0$, $\tilde{y}_0$ is a GHF of $(\tilde{y}_1, \tilde{y}_2)$,

$$\tilde{y}_0(\mu^{a_1}\tilde{y}_1, \mu^{a_2}\tilde{y}_2) = \mu\tilde{y}_0(\tilde{y}_1, \tilde{y}_2) , \tag{6.11}$$

i.e., $\tilde{y}_0 = \mathcal{F}_2(\tilde{y}_1, \tilde{y}_2)$ is an invariant equation under a group $\mathcal{G}_2$ defined by

$$\mathcal{G}_2 \begin{cases} \tilde{y}_0' = \mu\tilde{y}_0 , & (6.12a) \\ \tilde{y}_i' = \mu^{a_i}\tilde{y}_i , \quad i = 1, 2 . & (6.12b) \end{cases}$$

In general, the group $\mathcal{G}_2$ will be different (having different a_i) for *each* critical line at the tricritical point, and will only be valid within a certain region close to the critical line. The different groups $\mathcal{G}_2$ for the different lines L_i do not have regions of overlap and there is therefore no conflict.

We can now form absolute invariants of $\mathcal{G}_2$. Scaling with respect to the weak direction we obtain

$$z_0 \equiv \frac{\tilde{y}_0}{\tilde{y}_2^{1/a_2}} , \quad z_1 \equiv \frac{\tilde{y}_1}{\tilde{y}_2^{a_1/a_2}} . \tag{6.13}$$

Theorem 2 of Sec. V states that under the hypothesis (6.11), $\tilde{y}_0(\tilde{y}_1, \tilde{y}_2)$ may be expressed as[27]

$$z_0 = \mathcal{F}_1(z_1) . \tag{6.14}$$

The simplest example of this is for the line L_1 of Fig. 15. Here the variables of scaling are

$$\hat{y}_1 = \bar{x}_1 / \bar{x}_3^{\bar{a}_1/\bar{a}_3} , \tag{6.15a}$$

$$\hat{y}_2 = (\bar{x}_2 / \bar{x}_3^{\bar{a}_2/\bar{a}_3} + k) , \tag{6.15b}$$

where k is defined in Fig. 15. Rotation is not necessary and $\bar{R}_{ij} \equiv \delta_{ij}$. Hence on using (6.13) and (6.15), Eq. (6.14) can be written in the "double-power law" form[27]

$$\frac{G}{\bar{x}_3^{1/\bar{a}_3}(\bar{x}_2/\bar{x}_3^{\bar{a}_2/\bar{a}_3} + k)^{1/a_2}} = \mathcal{F}_1\left[\frac{\bar{x}_1}{\bar{x}_3^{\bar{a}_1/\bar{a}_3}(\bar{x}_2/\bar{x}_3^{\bar{a}_2/\bar{a}_3} + k)^{a_1/a_2}}\right] . \tag{6.16}$$

For a simple system with $n = 2$, scaling functions predict data collapsing for functions of two variables from a surface onto a line. For $n = 3$ and functions of three variables, data collapse from a volume onto a surface. However, the double-power scaling function of Eq. (6.16) predicts that data will collapse from a volume onto a line. Clearly this happens only within the region of validity of both groups of transformations $\mathcal{G}_2$ and $\mathcal{G}_3$.

The region of influence of $\mathcal{G}_2$ in the neighborhood of the tricritical point ${}^3R_0$ should also be controlled by the group $\mathcal{G}_3$. This means that the region of influence of $\mathcal{G}_2$ should be bounded by a surface which scales toward the ${}^3R_0$. In Fig. 15, where a line which scales is represented by a point, a surface which scales will be represented by a line. We therefore plot the surfaces bounding the region of influence of the group $\mathcal{G}_2$ (of transformations about a ${}^2R_1$, L_i) as a line surrounding the point in the y_1-y_2 plane, representing the particular line L_i.

In terms of the variables y_1, y_2 such a line will be represented by the equation

$$f(y_1, y_2) = 0 \tag{6.17a}$$

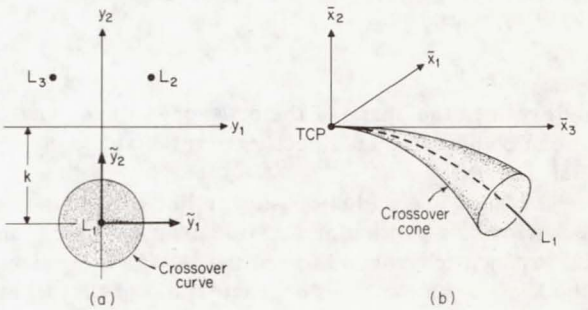

FIG. 15. (a) Plot of the invariants (y_1, y_2) for the group $\mathcal{G}_3$ of transformations about the tricritical point. The strong and weak directions for the line L_1 are y_1 and y_2. The circle around L_1 is a possible shape for the crossover region. (b) The principal points of interest of Fig. 15(a) in the full space $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$. The point labeled L_1 has become a line and the circle surrounding it has become a cone.

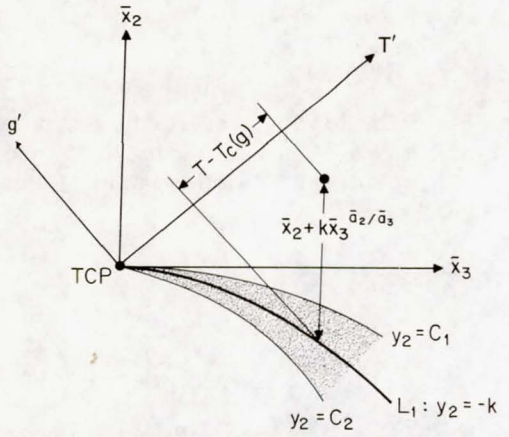

FIG. 16. Figure 15(b) sliced in the $\bar{x}_2$, $\bar{x}_3$ plane. The cone has become the two lines labeled $y_2 = C_1$, C_2 [see Fig. 15(a)]. These are generally referred to as crossover lines: $(T', g') = (T - T_t, g - g_t)$.

or

$$f(\bar{x}_1 / |\bar{x}_3|^{\bar{a}_1/\bar{a}_3}, \bar{x}_2 / |\bar{x}_3|^{\bar{a}_2/\bar{a}_3}) = 0 . \qquad (6.17b)$$

The area bounded by this curve (6.17) maps into a conical volume surrounding the critical line L_1 (Fig. 15). Scaling will not tell us the actual shape of the curve in the $y_1 - y_2$ plane but it does limit the shape in the $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ space, since all points in the y_1-y_2 plane give rise to curves approaching the tricritical point along a particular direction—the axis corresponding to the minimum $\bar{a}_i$.

In the plane $\bar{x}_1 = 0$, Eq. (6.16) requires[29] that

$$G \propto \bar{x}_3^{1/\bar{a}_3} (\bar{x}_2 / \bar{x}_3^{\bar{a}_2/\bar{a}_3} + k)^{1/a_2} , \qquad (6.18)$$

and the conical surface of Eq. (6.17) becomes the two "crossover lines"

$$\bar{x}_2 = C_{1,2} \bar{x}_3^{\bar{a}_2/\bar{a}_3} , \qquad (6.19)$$

as shown in Fig. 16. The crossover exponent φ,[7(a)] given by

$$\varphi \equiv \bar{a}_3 / \bar{a}_2 , \qquad (6.20)$$

determines the shape of the crossover lines, and it can be determined directly from the shape of the line L_1 ($^2R_1$).

The lines are called crossover lines, because the behavior of a particular function crosses over from an exponent characteristic of the $^3R_0$ far away from the line, to an exponent characteristic of the $^2R_1$ at points close to the $^2R_1$. It is important to realize that the group $\mathcal{G}_3$ does not cease to be valid, and the crossover does not refer to changing from one group to another. The group $\mathcal{G}_3$ is everywhere valid, and the crossover merely marks the limits of validity of $\mathcal{G}_2$. This principle will be extended in Paper II and the groups $\mathcal{G}_j$ will control crossovers (or boundaries for the regions of validity) of $\mathcal{G}_i$ where $j > i$.

Finally, a few remarks should be made about the exponents and the directions of approach to the tricritical point. Equation (6.18) shows that if $\bar{x}_2 / |\bar{x}_3|^{\bar{a}_2/\bar{a}_3}$ is a constant the exponent for G is $1/\bar{a}_3$. For a function f with a tricritical-point (TCP) scaling power $\bar{a}_f$, and a critical-line scaling power a_f, the equation analogous to Eq. (6.18) predicts exponents $\bar{a}_f/\bar{a}_3$ and a_f/a_2. For example, for the staggered susceptibility $\chi_{st} \equiv \partial^2 G / \partial H_{st}^2$, from (6.16),

$$\chi_{st} \propto \bar{x}_3^{(1-2\bar{a}_1)/\bar{a}_3} (\bar{x}_2 / \bar{x}_3^{\bar{a}_2/\bar{a}_3} + k)^{(1-2a_1)/a_2} , \qquad (6.21)$$

and $\bar{a}_f = 1 - 2\bar{a}_1$ and $a_f = 1 - 2a_1$ for this case. Thus for all the exponents considered below the numerators can be appropriately be replaced by $\bar{a}_f$ or a_f.

If the $^3R_0$ is approaches along a direction not asymptotically parallel to the $\bar{x}_3$ axis (i.e., outside the crossover lines), G scales with a power $1/\bar{a}_2$.

If the $^2R_1$ is approached along a line of constant $\bar{x}_3$, in the plane $\bar{x}_1 = 0$ Eq. (6.18) shows that G has an exponent $1/a_2$. This is expected, of course, since G has an exponent $1/a_2$ for any point (i.e., fixed x_3) on the line $^2R_1$ even when the point is far away from the $^3R_0$ [see Eq. (6.8)].

In Sec. V, exponents were demonstrated in terms of the scaling powers a_i [Eqs. (5.13)] and the same can be done here for the $^2R_1$ (exponents in terms of a_i) and the $^3R_0$ (exponents in terms of the $\bar{a}_i$) separately. The only new exponents, which will be derived, are exponents for the directions of approach to the $^3R_0$ along $y_2 =$ constant and these give exponents of the form

$$f \sim |x_3|^{\bar{a}_f/\bar{a}_3} . \qquad (6.22)$$

These can be related to exponents of approach along directions of type 2 by relations of the form

$$\bar{a}_f / \bar{a}_3 = (\bar{a}_f / \bar{a}_2)/\varphi . \qquad (6.23)$$

These are new predictions of scaling specific to tricritical points [the others are analogous to Eq. (5.13)].[7(c)]

Finally, we emphasize the importance of expressing the scaling relations in terms of invariants. For example, Eq. (6.21) may be written in the alternative "mixed-exponent form"

$$\chi_{st} \propto C(\bar{x}_2 + k\bar{x}_3^{1/\varphi})^{-\gamma} , \qquad (6.24)$$

where

$$C \propto \bar{x}_3^{(\gamma - \bar{\gamma})/\varphi} , \qquad (6.25)$$

with

$$-\gamma = \frac{1 - 2a_1}{a_2} , \qquad -\bar{\gamma} = \frac{1 - 2\bar{a}_1}{\bar{a}_2} . \qquad (6.26)$$

Expressions (6.24) and (6.25) appear more complicated than they actually are. The exponents are actually not mixed when expressed in the invariant from as shown in Eq. (6.21).

ACKNOWLEDGMENTS

We wish to thank F. Harbus, R. B. Griffiths, J. C. Wheeler, and E. K. Riedel for stimulating discussions.

APPENDIX A: PROOF OF THEOREM 1

Theorem 1. Consider a one-parameter continuous group of transformations

$$\mathcal{G}:\ x_i' = f_i(\lambda \,|\, x_0, x_1, \ldots, x_n), \tag{A1}$$

where $i = 0, 1, \ldots, n$. There exist n functionally independent absolute invariants of the x_i $(i = 0, 1, \ldots, n)$.

Proof of theorem 1. Consider a function $F(x_0', x_1', \ldots, x_n')$. Assume the derivatives of f_i with respect to λ exist. We expand $F(x_0', x_1', \ldots, x_n')$ in a Maclaurin series

$$F(x_0', x_1', \ldots, x_n') = f + f'(\delta\lambda) + \frac{f''}{2!}(\delta\lambda)^2 + \cdots, \tag{A2}$$

where

$$\begin{aligned} f &= F(x_0, x_1, \ldots, x_n), \\ f' &= \left(\frac{dF}{d\lambda}\right)_{\lambda=\lambda_0} = VF(x_0, x_1, \ldots, x_n), \\ f'' &= \left(\frac{d^2F}{d\lambda^2}\right)_{\lambda=\lambda_0} = V^2F(x_0, x_1, \ldots, x_n), \\ &\vdots \end{aligned} \tag{A3}$$

with λ_0 the value of λ corresponding to the identical transformation and

$$V \equiv \sum_{i=0}^{n} \xi_i \left(\frac{\partial}{\partial x_i}\right), \tag{A4}$$

$$\xi_i \equiv \left(\frac{\partial f_i}{\partial \lambda}\right). \tag{A5}$$

If, therefore, $F(x_0, x_1, \ldots, x_n)$ is an absolute invariant of the group $\mathcal{G}$, then

$$F(x_0', x_1', \ldots, x_n') = F(x_0, x_1, \ldots, x_n). \tag{A6}$$

The necessary and sufficient condition for $F = \text{const}$ is that

$$VF = \sum_{i=0}^{n} \xi_i \left(\frac{\partial F}{\partial x_i}\right) = 0. \tag{A7}$$

Thus, F is a solution of the partial differential equation (A7) and consequently, $F(x_0, x_1, \ldots, x_n) = \text{const}$ is a solution of the system of the equivalent ordinary differential equations

$$\frac{dx_0}{\xi_0} = \frac{dx_1}{\xi_1} = \cdots = \frac{dx_n}{\xi_n}. \tag{A8}$$

These equations admit n independent solutions (first integrals)[30,31] and theorem 1 is proved.

APPENDIX B: PROOF OF THEOREM 2

Theorem 2. If the equation

$$x_0 = X_0(x_1, x_2, \ldots, x_n) \tag{B1}$$

is invariant under

$$\mathcal{G}\begin{cases} x_0 = f(\lambda \,|\, x_0), \\ x_i' = f_i(\lambda \,|\, x_1, x_2, \ldots, x_n), \end{cases} \tag{B2}$$

then it can be expressed as

$$y_0 = Y_0(y_1, y_2, \ldots, y_{n-1}), \tag{B3}$$

where $(y_0, y_1, \ldots, y_{n-1})$ form a set of functionally independent absolute invariants under $\mathcal{G}$, with

$$\frac{\partial y_0}{\partial x_0} \neq 0,$$

and $(y_1, y_2, \ldots, y_{n-1})$, the $n-1$ functionally independent absolute invariants of $\mathcal{G}^s$.

By hypothesis, y_0 is an absolute invariant of $(x_0; x_1, \ldots, x_n)$ of $\mathcal{G}$. Thus, (B3) implies that the invariant equation (B1) may be written

$$y_0(x_0; x_1, x_2, \ldots, x_n) = Y_0(y_1, y_2, \ldots, y_{n-1}). \tag{B4}$$

Since $(y_1, y_2, \ldots, y_{n-1})$ form a basis set of functionally independent absolute invariants of $(x_1, x_2, \ldots, x_n)$ of $\mathcal{G}^s$, (B4) is equivalent to the statement that x_0 is expressible as an implicit function of $(x_1, x_2, \ldots, x_n)$:

$$y_0(x_0; x_1, x_2, \ldots, x_n) = g(x_1, x_2, \ldots, x_n), \tag{B5}$$

where g is an absolute invariant of $(x_1, x_2, \ldots, x_n)$ under $\mathcal{G}^s$.

Before we launch into the proof of the theorem, we give a proof of the following lemma.[32]

Lemma. A necessary and sufficient condition for x_0, implicitly defined by (B5), as a function of $(x_1, x_2, \ldots, x_n)$, to be the same function as x_0' of $(x_1', x_2', \ldots, x_n')$ implicitly defined by

$$y_0(x_0', x_1', x_2', \ldots, x_n') = g'(x_1', x_2', \ldots, x_n') \tag{B6}$$

is that g is an absolute invariant of $(x_1, x_2, \ldots, x_n)$ under $\mathcal{G}^s$.

Proof of lemma. Since y_0 is an absolute invariant of $\mathcal{G}$, we have

$$y_0(x_0; x_1, x_2, \ldots, x_n) = y_0(x_0'; x_1', x_2', \ldots, x_n'). \tag{B7}$$

Since we require $x_0(x_1, x_2, \ldots, x_n)$ to be exactly the same function as $x_0'(x_1', x_2', \ldots, x_n')$, (B5)–(B7) require that

$$\begin{aligned} g(x_1, x_2, \ldots, x_n) &= g'(x_1', x_2', \ldots, x_n') \\ &= g(x_1', x_2', \ldots, x_n'). \end{aligned} \tag{B8}$$

This is the necessity proof.

We now demonstrate (B8) is sufficient to ensure (B5) and (B6) admit an invariant solution such that

$x_0(x_1, x_2, \ldots, x_n)$ is exactly the same function as $x_0'(x_1', x_2', \ldots, x_n')$. Inverting (B5) and (B6), we obtain

$$x_0(x_1, x_2, \ldots, x_n) = h(g; x_1, x_2, \ldots, x_n),$$
$$x_0'(x_1', x_2', \ldots, x_n') = h'(g'; x_1, x_2, \ldots, x_n), \tag{B9}$$

in some neighborhood of the $(x_1, x_2, \ldots, x_n)$ and $(x_1', x_2', \ldots, x_n')$ spaces, respectively. It is obvious that $h(g; x_1, x_2, \ldots, x_n)$ is exactly the same function as $h'(g_1'; x_1', x_2', \ldots, x_n')$. But, by hypothesis, $g(x_1, x_2, \ldots, x_n)$ is exactly the same function as $g'(x_1', x_2', \ldots, x_n')$. Therefore, $x_0(x_1, x_2, \ldots, x_n)$ is exactly the same function as $x_0'(x_1', x_2', \ldots x_n')$.

Proof of theorem 2. Using the lemma and the fact that $g(x_1, x_2, \ldots, x_n)$ in (B5) is an absolute invariant of $(x_1, x_2, \ldots, x_n)$ under $\mathcal{G}^s$, we immediately verify the statement of theorem 2.

APPENDIX C: PROOF OF THEOREM 3

Theorem 3. Consider a one-parameter continuous group of transformations $\mathcal{G}$:

$$\mathcal{G}\begin{cases} x_0' = \lambda x_0, \\ x_i' = f_i(\lambda) x_i, \quad i = 1, 2, \ldots, n. \end{cases} \tag{C1}$$

If

$$x_0 = F(x_1, x_2, \ldots, x_n) \tag{C2}$$

is an invariant equation under $\mathcal{G}$, the most general form for $f_i(\lambda)$ is λ^{a_i}, where a_i are constants.

Proof of theorem 3. We transform (C2) by means of (C1) for two successive values of $\lambda = \lambda_1$, λ_2,

$$\lambda_1\lambda_2 x_0 = F[f_1(\lambda_1)f_1(\lambda_2)x_1, f_2(\lambda_1)f_2(\lambda_2)x_2, \ldots, f_n(\lambda_1)f_n(\lambda_2)x_n], \tag{C3}$$

and again for the value of $\lambda = \lambda_1\lambda_2$,

$$\lambda_1\lambda_2 x_0 = F[f_1(\lambda_1\lambda_2)x_1, f_2(\lambda_1\lambda_2)x_2, \ldots, f_n(\lambda_1\lambda_2)x_n]. \tag{C4}$$

These results are to hold for all values of x_i. Setting $x_2 = x_3 = \cdots = x_n = 0$, we have from (C3) and (C4)

$$F[f_1(\lambda_1)f_1(\lambda_2)x_1, 0, \ldots, 0] = F[f_1(\lambda_1\lambda_2)x_1, 0, \ldots, 0]. \tag{C5}$$

Therefore,

$$f_1(\lambda_1)f_1(\lambda_2) = f_1(\lambda_1\lambda_2). \tag{C6}$$

The solution[33] of the functional equation for $f_1(\lambda)$ for $\lambda > 0$ is

$$f_1(\lambda) = \lambda^{a_1}, \tag{C7}$$

where a_1 is a constant. This process may be repeated for each $f_1(\lambda)$, $i = 1, 2, \ldots, n$. Thus, theorem 3 is proved.

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*Work forms a part of a Ph.D. thesis to be submitted to the Physics Department of MIT by A. Hankey. Preliminary reports of aspects of this work appear in T. S. Chang, A. Hankey, and H. E. Stanley, AIP Conf. Proc. 10, 880 (1973).

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§Supported by NSF, ONR. and AFOSR

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where; a preliminary account appears in A. Hankey, T. S. Chang, and H. E. Stanley, AIP Conf. Proc. **10**, 899 (1973).

[17]This point occurs when $2mH = 2J_{SR}$, where $2m = 1$, in the remainder of this work and J_{SR} is the nearest-neighbor exchange energy [cf. Eq. (3.1)].

[18]The inclusion of a competing long-range interaction makes it possible to define a fourth variable, which is fieldlike, although it is strictly not a field. This variable is the ratio of interaction strengths in the Hamiltonians. In the figures, it is denoted by $\tilde{\mathcal{R}} \equiv J_{SR}/J_{LR}$. This variable is "fieldlike" because the CXS and CRS in the space including $\tilde{\mathcal{R}}$ have the same properties as in spaces of fields.

[19]An exact analysis of an Ising model with several staggered fields requires consideration of a regularly repeating block of spins. The length of the block will be equal to the lowest common multiple of the wavelengths of the various staggered fields. If staggered fields of wavelengths 2 and 3 had been chosen, there would be six spins in a block, but with wavelengths 2 and 4, there are only four spins per block. The analysis for a block of n spins proceeds by multiplying together n transfer matrices and so H_2 and H_4 give the smallest number of matrices to multiply.

[20]Our ideas of using the invariant theorems of continous groups of transformations for multicomponent scaling were first presented at the MIT Summer School on Critical Phenomena in 1971 (unpublished). Recently, important advances have been made by Wilson, Fisher, Wegner, Riedel and others in generating the scaling exponents using the linearized renormalization group equations. It was shown that the scaling fields may be deduced from the relevant operator densities. The number of references to work utilizing the renormalization group now exceeds 100, and the reader is urged to consult K. G. Wilson, Phys. Reports (to be published) or *Cooperative Phenomena Near Phase Transitions: A Bibliography with Selected Readings*, edited by H. E. Stanley (MIT Press, Cambridge, Mass., 1973).

[21]This form of the scaling hypothesis has been treated in detail by A. Hankey and H. E. Stanley [Phys. Rev. B **6**, 3515 (1972)]. It is a more complete discussion of the statements of the hypothesis previously used by Riedel and Wegner (Refs. 3 and 4) and also by Kadanoff (Ref. 2c). For a complete set of references, see *Cooperative Phenomena near Phase Transitions: A Bibliography with Selected Readings,* edited by H. E. Stanley (MIT Press, Cambridge, Mass., 1973).

[22]The elements of the group are parametrized by the single parameter λ. Thus the group is a one-parameter group.

[23]A more-general linear transformation will be $x_i = f_i(\lambda)x_i$. Because of the associativity property of groups, it can be shown that $f_i(\lambda) = \lambda^{a_i}$ [Appendix C]. If some of the x_i do not scale and serve only as "inactive parameters" in Eq. (5.6), the corresponding scaling powers are equal to zero.

[24]We could, of course, replace H_{st} by x_1 in Eq. (5.9), but we emphasize that (cf. Fig. 12) the only thermodynamic field axis which is a strong direction is H_{st}. Thus while the direction $H_{st} + \epsilon(H - H_c)$ or $H_{st} + \epsilon(T - T_c)$ are both strong and $\partial/\partial H_{st}$ is the same as $\partial/\partial x_1$, $\partial/\partial T$, and ∂/∂H are only equivalent to differentiating with respect to x_2.

[25]Here we will formulate the scaling hypothesis only for the Gibbs function, because the scaling equation analogous to (5.1) for other functions can be obtained by appropriate differentation with respect to its independent variables. Scaling functions for physcially more interesting functions like M and χ can then be obtained by the same method used here for G.

[26]Equations (5.14) can all be written in terms of $|H - H_c(T)|$ except at the Néel point, where the line of critical points is parallel to the H axis.

[27]Occasionally, it turns out to be appropriate to scale with respect to a variable which is not tangent—see, e.g., Paper II. This is also true when all the ${}^2R_1$ do not end up parallel and the tangent variable for one line is *not* the tangent variable for another line. We emphasize that scaled equations may change their values or functional forms when the scaling variables change signs.

[28]"Mutually consistent" means that if one chooses two different sets of axes for scaling, one set $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ with reference to a line L and the second $(\bar{x}_1', \bar{x}_2', \bar{x}_3')$ with repsect to L', then the second set is none other than a permutation of the first set.

[29]Our expression differs from that given in Ref. 7 [Eq. (8)], where the constant k is not present. The constant k is needed to ensure the divergence property of χ_{st} (and other second derivatives of G) along the critical line.

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PHYSICAL REVIEW B VOLUME 7, NUMBER 9 1 MAY 1973

Double-Power Scaling Functions near Tricritical Points*

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(Received 11 September 1972)

We introduce invariants of the scaling equation about the tricritical point. Using these invariants, a modified version of the scaling hypothesis about the three critical lines meeting at the tricritical point is presented. From it we demonstrate that the thermodynamic equation of state near a tricritical point and near a critical line may be expressed as double-power scaling functions. These imply that experimental data should collapse from a *volume* onto a *line* (i.e., by two dimensions). This behavior is in contrast to ordinary "single-power" scaling functions, which predict data collapsing from a volume onto a surface or from a surface onto a line (i.e., by one dimension).

I. INTRODUCTION

There have been recent experimental measurements[1] near tricritical points[2] (TCPs) in metamagnets, NH_4Cl, and ^3He-^4He mixtures.[1] These data have been partially interpreted recently in terms of scaling arguments in which one makes not one but two scaling hypotheses.[3-5] Riedel and Wegner[6] were perhaps the first authors to note that in regions for which two scaling hypotheses are simultaneously valid, double-power-law behavior of certain functions results. In this work we present a *variation* of scaling for tricritical points, utilizing generalized homogeneous functions[7] (GHF's) of invariants of the scaling equation about the tricritical point. We obtain, in regions near a critical line and a tricritical point, double-power scaling functions which permit data to collapse from a volume onto a line, in contrast to the behavior of single-power scaling functions, which permit data to collapse by only one dimension (e.g., from a surface onto a line, or a volume onto a surface).

Before we can proceed to make the scaling hypothesis for a TCP, it will be necessary to determine the relevant directions for scaling.[3] The three thermodynamic fields (T, temperature; η, ordering field; and g, nonordering field) near a TCP are believed to constitute an affine space in which directions may be defined by parallelism only. A TCP is a point of intersection of three critical lines in this three-dimensional affine space (cf. Fig. 1). At each point P on a critical line, three different types of directions can be established. The first direction, $x_1(P)$, is a direction not locally parallel to the coexistence surface. The second, $x_2(P)$, is locally parallel to the coexistence surface but not parallel to the critical line. These are the "strong" and "weak" directions of Griffiths and Wheeler.[8] The third direction, $x_3(P)$, is locally parallel to the critical line.

As the point P moves toward the TCP, these directions attain limiting orientations. Since there are three critical lines terminating at a TCP, three "rival" sets of directions of this type exist at the TCP. It has been shown[4] that if scaling holds at a TCP, these three sets of directions are equivalent. Thus, we choose the relevant directions for scaling at a TCP as $\bar{x}_i \equiv \lim(P \to \mathrm{TCP}) x_i(P)$, where P is a point on the critical line L_1 (see Fig. 1).[9]

II. SCALING HYPOTHESIS FOR TCP

Having ascertained the relevant scaling directions $\bar{x}_i$ for TCP, we now introduce a scaling parameter λ (>0) and make the homogeneity hypothesis [3-7] that the singular part of the Gibbs potential is asymptotically a GHF,

$$G(\lambda^{\bar{a}_1}\bar{x}_1,\ \lambda^{\bar{a}_2}\bar{x}_2, \lambda^{\bar{a}_3}\bar{x}_3) = \lambda G(\bar{x}_1,\ \bar{x}_2,\ \bar{x}_3)\,, \tag{1}$$

where $\bar{a}_i$ are the scaling powers. Equation (1) is *equivalent* to the statement that

$$G = F_3\,(\bar{x}_1,\ \bar{x}_2,\ \bar{x}_3) \tag{2}$$

is an invariant equation under the one-parameter (λ) group ($\mathcal{G}_3$) of transformations

$$G' = \lambda G,\quad \bar{x}_i' = \lambda^{\bar{a}_i}\bar{x}_i,\quad (i = 1,\ 2,\ 3)\,. \tag{3}$$

In other words, under the transformations, Eq. (2) becomes $G' = F_3(\bar{x}_1',\ \bar{x}_2',\ \bar{x}_3')$.

The group $\mathcal{G}_3$ admits a basis set of three ($i = 0$, 1, 2) functionally independent absolute invariants, $y_i(G', \bar{x}_1',\ \bar{x}_2',\ \bar{x}_3') = y_i(G, \bar{x}_1,\ \bar{x}_2,\ \bar{x}_3)$, such that all

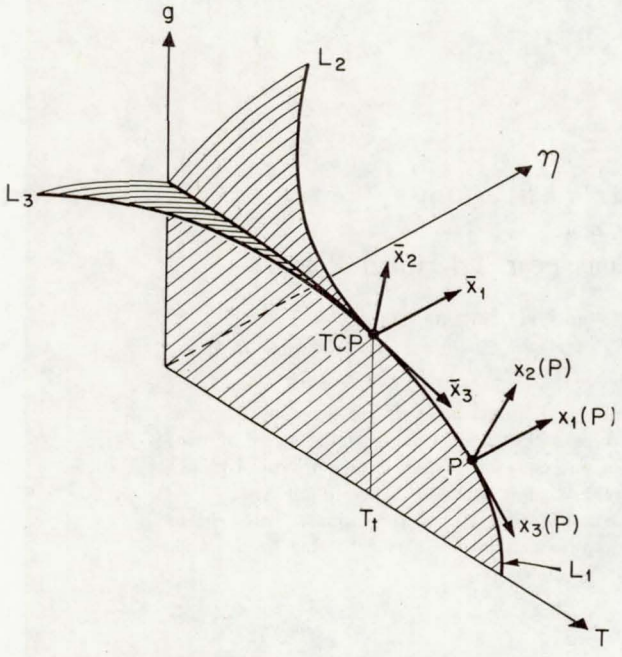

FIG. 1. Schematic phase diagram showing a TCP (at $T=T_t$). Shaded areas are coexistence surfaces. At a point P on L_1, a triad of directions $x_i(P)$ are shown. This triad becomes $\bar{x}_i$ at TCP.

other absolute invariants are expressible in terms of these. One such basis set is

$$y_0 \equiv G/\bar{x}_3^{1/\bar{a}_3},\quad y_1 \equiv \bar{x}_1/\bar{x}_3^{\bar{a}_1/\bar{a}_3},\quad y_2 \equiv \bar{x}_2/\bar{x}_3^{\bar{a}_2/\bar{a}_3}\ . \tag{4}$$

The scaling hypothesis, Eq. (1), requires Eq. (2) to be expressible in terms of the basis set as a "single-power" scaling function,

$$y_0 = \bar{F}_2(y_1,\ y_2)\ , \tag{5}$$

which states that G (and other thermodynamic functions), when appropriately scaled, are functions of the invariants $(y_1,\ y_2)$ alone. This result allows data near a TCP to collapse from a *volume* onto a *surface*.

We remark that, using Eq. (1), it is possible to determine all exponent relations and "single-power" scaling laws for a TCP.[3–5]

III. GEOMETRY OF SURFACES AND CURVES NEAR TCP

Since the quantities y_1 and y_2 defined in Eqs. (4) form a basis set of functionally independent absolute invariants of $\bar{x}_i$ under the group of transformations $\bar{x}_i' = \lambda^{\bar{a}_i}\bar{x}_i$, *points* in the invariant $(y_1,\ y_2)$ plane give rise to invariant *curves* in the $(\bar{x}_1,\ \bar{x}_2,\ \bar{x}_3)$ space. We have seen that the scaling hypothesis requires scaled thermodynamic functions near a TCP to depend on y_1 and y_2 only. This implies that each of the three critical *lines* near the TCP can be expressed as a *point* $y_i = k_i$ in the (y_1, y_2) plane, where k_i are constants.

Usually, for systems exhibiting a TCP, one of the critical lines is a planar curve lying entirely in the $(g,\ T)$ plane (e.g., L_1 of Fig. 1). Since $\bar{x}_1 = 0$, Eq. (4) implies that L_1 is given by $(y_1,\ y_2) = (0,\ -k)$ in the invariant plane.

Near L_1, it is expected that the symmetry property of the critical line will also influence the asymptotic form of the thermodynamic functions. The region of influence is bounded by some "crossover" curve, $f_x(y_1,\ y_2) = 0$ [Fig. 2(a)], or

$$f_x(x_1/x_3^{\bar{a}_1/\bar{a}_3},\ \bar{x}_2/\bar{x}_3^{\bar{a}_2/\bar{a}_3}) = 0, \tag{6}$$

which is a conical surface surrounding L_1 in the $(\bar{x}_1,\ \bar{x}_2,\ \bar{x}_3)$ space [Fig. 2(b)]. Scaling cannot tell us the *actual* shape of the curve in the $(y_1,\ y_2)$ plane,[10] but it does limit the shape of the conical "crossover" surface in the $(\bar{x}_1,\ \bar{x}_2,\ \bar{x}_3)$ space, since all points in the $(y_1,\ y_2)$ plane give rise to curves approaching the TCP along the $\bar{x}_3$ axis (corresponding to the minimum $\bar{a}_i$).[11]

IV. DOUBLE-POWER SCALING FUNCTIONS FOR L_1

We now proceed to deduce the restriction on the asymptotic form of the thermodynamic functions near a TCP adjacent to the critical line L_1.[12] Along L_1, the conventional scaling hypothesis is

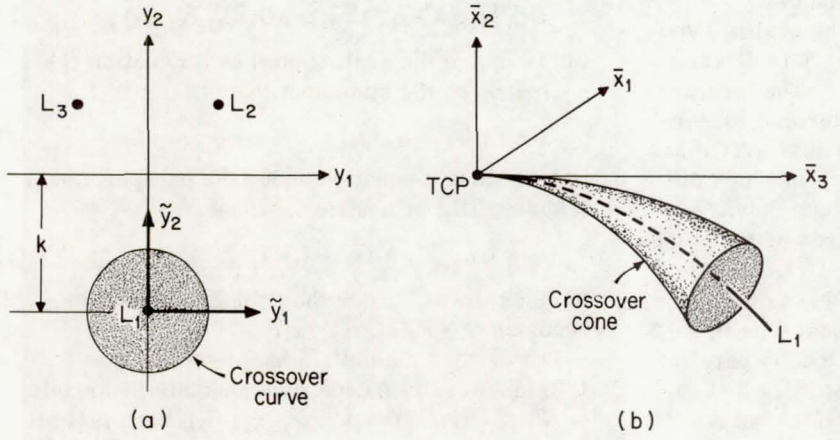

FIG. 2. (a) Invariant $(y_1,\ y_2)$ plane. The strong and weak directions for L_1 are y_1 and y_2, and the crossover curve is shown (Ref. 8). (b) Principal points of interest of (a) in the $(\bar{x}_1,\ \bar{x}_2,\ \bar{x}_3)$ space.

normally stated in terms of a GHF equation

$$G(\mu^{a_1} x_1,\ \mu^{a_2} x_2;\ x_3) = \mu G(x_1,\ x_2;\ x_3), \tag{7}$$

where $\mu(>0)$ is an arbitrary parameter, a_1 and a_2 are the scaling powers for L_1, and x_3 is an "inactive" variable which does not scale. Near the TCP, by Eq. (5) the value of $y_0 \equiv G/\bar{x}_3^{1/\bar{a}_3}$ changes only if the values y_1 and/or y_2 change. It is much better therefore to make a scaling hypothesis about L_1 near the TCP using y_1 and y_2.[13]

Since the coexistence surface bounded by L_1 maps into the vertical axis of the $(y_1,\ y_2)$ plane, the direction y_1 is strong and y_2 is weak. Thus, we deduce that the proper scaling variables for L_1 (near the TCP) are

$$\tilde{y}_1 \equiv y_1,\quad \tilde{y}_2 \equiv y_2 + k; \tag{8}$$

these vanish at the line $(y_1,\ y_2) = (0,\ -k)$.

We now hypothesize that along L_1 near the TCP, $\tilde{y}_0 \equiv y_0$ is a GHF of $(\tilde{y}_1,\ \tilde{y}_2)$:

$$\tilde{y}_0(\mu^{a_1}\tilde{y}_1,\ \mu^{a_2}\tilde{y}_2) = \mu\tilde{y}_0(\tilde{y}_1,\ \tilde{y}_2). \tag{9}$$

In other words, instead of (7) we postulate that

$$\tilde{y}_0 = F_2(\tilde{y}_1,\ \tilde{y}_2) \tag{10}$$

is an invariant equation under the group $(\mathcal{G}_2)$ of transformations $(\tilde{y}_0' = \mu\tilde{y}_0, \tilde{y}_1' = \mu^{a_1}\tilde{y}_1$, and $\tilde{y}_2' = \mu^{a_2}\tilde{y}_2)$.

By the same reasoning used for the derivation of Eq. (5), we see that (9) requires that (10) may be written in the form

$$z_0 = F_1(z_1), \tag{11}$$

where $z_0 \equiv \tilde{y}_0/\tilde{y}_2^{1/a_2}$ and $z_1 \equiv \tilde{y}_1/\tilde{y}_2^{a_1/a_2}$ form a set of functionally independent absolute invariants of $\mathcal{G}_2$ and therefore of the variables G, $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$ under the direct-product group $\mathcal{G}_2 \otimes \mathcal{G}_3$.

Reexpressing Eq. (11) in terms of the original variables G, x_1, x_2, and x_3 we obtain the "double-power" form[14]

$$\frac{G}{\bar{x}_3^{1/\bar{a}_3}(\bar{x}_2/\bar{x}_3^{\bar{a}_2/\bar{a}_3} + k)^{1/a_2}} = F_1\left(\frac{\bar{x}_1}{\bar{x}_3^{\bar{a}_1/\bar{a}_3}(\bar{x}_2/\bar{x}_3^{\bar{a}_2/\bar{a}_3} + k)^{a_1/a_2}}\right). \tag{12}$$

Equation (12) predicts that near the TCP *and* L_1, data will collapse from a *volume* onto a *line*. Clearly, this happens only within the crossover cone of Eq. (6) [cf. Fig. 2(b)].

In the plane $\bar{x}_1 = 0$ [i.e., the (g, T) plane], Eq. (12) requires[15]

$$G \sim \bar{x}_3^{1/\bar{a}_3}(\bar{x}_2/\bar{x}_3^{\bar{a}_2/\bar{a}_3} + k)^{1/a_2}, \tag{13}$$

and the conical surface of Eq. (6) becomes two crossover lines (cf. Fig. 3) $\bar{x}_2 = C_i\bar{x}_3^{\bar{a}_2/\bar{a}_3}$ or $y_2 = C_i$, where $i = 1, 2$. The crossover exponent $\varphi \equiv \bar{a}_3/\bar{a}_2$, which determines the shape of the crossover lines, can be obtained directly from the shape of L_1.[3]

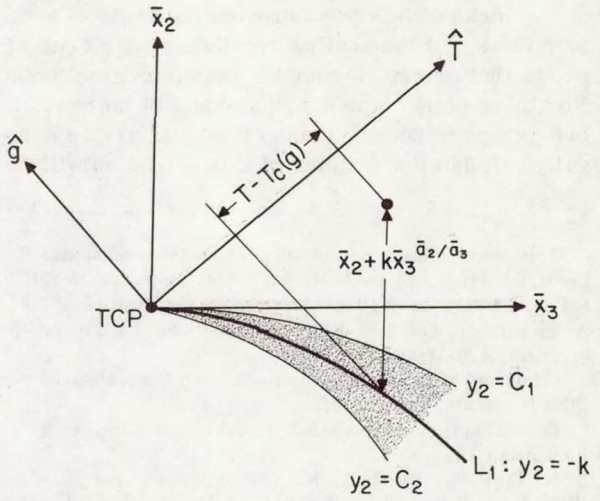

FIG. 3. Figure 2(b) sliced in the $\bar{x}_1 = 0$ plane. The crossover lines are labeled $y_2 = C_1,\ C_2$. The projection of $\bar{x}_2 + k\bar{x}_3^{\bar{a}_2/\bar{a}_3}$ along the T axis is $T - T_c(g)$, and $(\hat{T},\ \hat{g}) \equiv (T - T_t,\ g - g_t)$.

V. EXTENSIONS AND CONCLUDING REMARKS

The entire discussion in this paper may be extended to the scaling of any thermodynamic function f. For example, for the staggered susceptibility $\chi_{st} \equiv \partial^2 G/\partial\eta^2 \propto \partial^2 G/\partial\bar{x}_1^2$ (or $\partial^2 G/\partial x_1^2$) and the direct susceptibility $\chi \equiv \partial^2 G/\partial g^2 \propto \partial^2 G/\partial\bar{x}_2^2$ (or $\partial^2 G/\partial x_2^2$) of a metamagnet, the expression analogous to Eq. (13) is

$$f_i = \bar{x}_3^{(1-2\bar{a}_i)/\bar{a}_3}(\bar{x}_2/\bar{x}_3^{\bar{a}_2/\bar{a}_3} + k)^{(1-2a_i)/a_2}, \tag{14}$$

where $f_1 \equiv \chi_{st}$ and $f_2 \equiv \chi$. We note that Eq. (13) has the appropriate divergence properties at the critical line *and* at the TCP.

Finally, we make a few remarks about the exponents and directions of approach toward the TCP and L_1. Using the experimentally accessible function χ_{st} as an example, we note that if we approach the TCP along a curve $\bar{x}_2/\bar{x}_3^{\bar{a}_2/\bar{a}_3} = \text{const}$, the scaling exponent is $-\bar{\gamma}_{st}\varphi^{-1} = (1 - 2\bar{a}_1)/\bar{a}_3$. If we approach the critical line L_1 along a line $\bar{x}_3 = \text{const}$, the scaling exponent is $-\gamma_{st} = (1 - 2a_1)/a_2$ as expected. On the other hand, if the TCP is approached along a path outside the crossover lines, χ_{st} scales with an exponent $-\bar{\gamma}_{st} = (1 - 2\bar{a}_1)/\bar{a}_2$. Similar remarks may be made with respect to the three-dimensional "double-power" scaling functions of Eq. (12).

Equation (14) may be cast in "mixed-exponent" forms; e.g., $\chi_{st} \sim C[T - T_c(g)]^{-\gamma_{st}}$, in which $\bar{x}_2 + k\bar{x}_3^{\bar{a}_2/\bar{a}_3}$ has been replaced by its projection along the T axis (Fig. 3), and $C \sim \bar{x}_3^{(\gamma_{st} - \bar{\gamma}_{st})/\varphi}$ is the asymptotic amplitude. Depending on the relative magnitudes of γ_{st} and $\bar{\gamma}_{st}$, the asymptotic amplitude may diverge, vanish, or stay constant as the TCP is approached within the crossover cone.

The ideas of this work provide the basis of a formulation of the scaling hypothesis near critical points that are more complex than tricritical points. For these points, the direct product of more than two groups of scaling transformations arises naturally. A detailed account of this extension will be published elsewhere.

ACKNOWLEDGMENTS

The authors wish to thank F. Harbus, R. B. Griffiths, E. K. Riedel, and J. C. Wheeler for stimulating discussions.

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*This work forms a portion of a Ph.D. thesis submitted to the MIT Physics Dept. by A. Hankey. Work supported by National Science Foundation, National Aeronautics and Space Administration, Office of Naval Research, and Air Force Office of Scientific Research.

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[9]In general, the $\bar{x}_j$ may be nonorthogonal, but in this work we need consider only rectilinear orthogonal coordinates.

[10]Although the crossover curve may be of any form, it is shown for convenience as a circle in Fig. 2(a).

[11]In this paper, we assume $a_1 \neq a_2 \neq a_3$. (See Ref. 4 for discussion of other cases.)

[12]The scaling ideas for $L_{2,3}$ are similar. However, it will be necessary to rotate the invariant axis to determine the strong and weak directions.

[13]This assumption is equivalent to the scaling assertion of Ref. 6, that the extended homogeneity relation (1) must encompass the relation (7) asymptotically close to the critical line.

[14]Equation (12) is similar to that found in Ref. 6(b) for dynamic scaling in anisotropic magnetic systems. We emphasize that scaled equations may change their values or functional forms when the scaling variables change signs.

[15]Equation (13) is essentially equivalent to Eq. (8) of Ref. 3, provided the variable μ_1 is interpreted as the critical line scaling field.